Measure and Integration: Final Exam 2021-22

Exam is open book, but only the book of R. Schilling: Measures, Integrals and Martingales is allowed

(1) Consider the measure space \(([0, 1], \mathcal{B}([0, 1]), \lambda)\), where \(\mathcal{B}([0, 1])\) and \(\lambda\) are the restrictions of the Borel \(\sigma\)-algebra and Lebesgue measure to \([0, 1]\). Let \(u \in \mathcal{M}(\mathcal{B}([0, 1]))\) be a bounded function that is continuous \(\lambda\) almost everywhere. Define \(u_n(x) = u\left(\frac{n x}{n + 1}\right)\) for \(n \geq 1\). Prove that for any \(p \in [1, \infty)\) one has
\[
\lim_{n \to \infty} \|u_n\|_p = \|u\|_p.
\]
(1.5 pt)

(2) Consider the measure space \(((0, \infty), \mathcal{B}((0, \infty)), \lambda)\), where \(\mathcal{B}((0, \infty))\) is the Borel \(\sigma\)-algebra restricted to \((0, \infty)\) and \(\lambda\) is the restriction of Lebesgue measure to \((0, \infty)\). Let \(u(x) = \frac{x}{e^x - 1}\) for \(x \in (0, \infty)\).

(a) Prove that \(u \in \mathcal{L}^1((0, \infty), \lambda)\). (Hint: \(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\) ) (1.5 pts)

(b) Prove that \(\lim_{n \to \infty} \int_{(0, \infty)} \frac{n}{e^x - 1} \sin\left(\frac{x}{n}\right) d\lambda(x) = \int_{(0, \infty)} u(x) d\lambda(x)\) . (Hint: \(|\frac{\sin y}{y}| \leq 1\) for all \(y\) and \(\lim_{y \to 0} \frac{\sin y}{y} = 1\) ) (1.5 pts)

(3) Let \((X, \mathcal{A}, \mu)\) be a measure space, and \(u \in \mathcal{M}(\mathcal{A})\) satisfies \(u^n \in \mathcal{L}(\mu)\) for all \(n \geq 1\). Assume \(\int u^{2n} d\mu = c\) and \(\int u^{2n-1} d\mu = d\) for all \(n \geq 1\), where \(c\) and \(d\) are some constants.

(a) Prove that \(u\) takes three distinct values 0, 1 and \(-1\) \(\mu\) a.e. (Hint: consider the function \(u^2(u-1)^2(u+1)^2\)) (1 pt)

(b) Let \(A = \{ x \in X : u(x) = 1 \} \) and \(B = \{ x \in X : u(x) = -1 \} \). Prove that \(c \geq d\) and that \(\mu(A) = \frac{c+d}{2}\) and \(\mu(B) = \frac{c-d}{2}\). (1.5 pt)

(4) Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) be \(\sigma\)-finite measure spaces. Suppose \(f \in \mathcal{L}^1(\mu)\) and \(g \in \mathcal{L}^1(\nu)\) are non-negative. Define measures \(\mu_2\) on \(\mathcal{A}\) and \(\nu_2\) on \(\mathcal{B}\) by
\[
\mu_2(A) = \int_A f \, d\mu_1 \quad \text{and} \quad \nu_2(B) = \int_B g \, d\nu_1,
\]
for \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\).

(a) Prove that if \(v \in \mathcal{M}_\mathbb{R}(\mathcal{B})\), then \(\int_Y v \, d\nu_2 = \int_Y vg \, d\nu_1\). (1.5 pt)

(b) Prove that for every \(D \in \mathcal{A} \otimes \mathcal{B}\) one has
\[
(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y) \, d(\mu_1 \times \nu_1)(x, y).
\]
(1.5 pt)