Measure and Integration: Final Exam 2021-22
Exam is open book, but only the book of R. Schilling: Measures, Integrals and Martingales is allowed

(1) Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to $[0,1]$. Let $u \in \mathcal{M}(\mathcal{B}([0,1]))$ be a bounded function that is continuous λ almost everywhere. Define $u_n(x) = u\Big(\dfrac{nx}{n+1}\Big)$ for $n \geq 1$. Prove that for any $p \in [1,\infty)$ one has
$$\lim_{n\to\infty} ||u_n||_p = ||u||_p.$$
(1.5 pt)

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty)), \lambda)$, where $\mathcal{B}((0,\infty))$ is the Borel σ-algebra restricted to $(0,\infty)$ and λ is the restriction of Lebesgue measure to $(0,\infty)$. Let $u(x) = \dfrac{x}{e^x - 1}$ for $x \in (0,\infty)$.

(a) Prove that $u \in \mathcal{L}^1((0,\infty), \lambda)$. (Hint: $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$) (1.5 pts)

(b) Prove that $\lim_{n\to\infty} \int_{(0,\infty)} \dfrac{n}{e^x - 1} \sin(\dfrac{x}{n})\, d\lambda(x) = \int_{(0,\infty)} u(x)\, d\lambda(x)$.(Hint: $|\dfrac{\sin y}{y}| \leq 1$ for all y and $\lim_{y\to 0} \dfrac{\sin y}{y} = 1$.) (1.5 pts)

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{M}(\mathcal{A})$ satisfies $u^n \in \mathcal{L}(\mu)$ for all $n \geq 1$. Assume $\int u^{2n}\, d\mu = c$ and $\int u^{2n-1}\, d\mu = d$ for all $n \geq 1$, where c and d are some constants.

(a) Prove that u takes three distinct values 0, 1 and -1 μ a.e. (Hint: consider the function $u^2(u-1)^2(u+1)^2$.) (1 pt)

(b) Let $A = \{x \in X : u(x) = 1\}$ and $B = \{x \in X : u(x) = -1\}$. Prove that $c \geq d$ and that $\mu(A) = \dfrac{c+d}{2}$ and $\mu(B) = \dfrac{c-d}{2}$. (1.5 pt)

(4) Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be σ-finite measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are **non-negative**. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by
$$\mu_2(A) = \int_A f\, d\mu_1 \text{ and } \nu_2(B) = \int_B g\, d\nu_1,$$
for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) Prove that if $v \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{B})$, then $\int_Y v\, d\nu_2 = \int_Y vg\, d\nu_1$. (1.5 pt)

(b) Prove that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has
$$(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y)\, d(\mu_1 \times \nu_1)(x,y).$$
(1.5 pt)