**Measure and Integration: Solution Retake Exam 2021-22**

(1) Let $(X,\mathcal{A},\mu)$ be a measure space and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$-measurable transformation. Let
$$\mathcal{D} = \{A \in \mathcal{A} : \mathbb{I}_A = \mathbb{I}_{T^{-1}A} \ \ \mu \text{ a.e.}\}.$$

(a) Prove that $\mathcal{D}$ is a Dynkin system. (1.5 pts)

(b) Prove that $\mathcal{D}$ is $\bigcap$-stable. (1 pt)

(c) Prove that $\mathcal{D}$ is a $\sigma$-algebra. (0.5 pts)

**Proof (a)**: We verify the three conditions of a Dynkin system. Since $X = T^{-1}X$, we see that $\mathbb{I}_X(x) = \mathbb{I}_{T^{-1}X}(x)$ for all $x \in X$, hence $X \in \mathcal{D}$. Let $A \in \mathcal{D}$ and $N = \{x \in X : \mathbb{I}_A(x) \neq \mathbb{I}_{T^{-1}A}(x)\}$, then $\mu(N) = 0$. Note that
$$T^{-1}A^c = T^{-1}X \setminus T^{-1}A = X \setminus T^{-1}A = (T^{-1}A)^c.$$
Thus, for $x \in N^c$ we have
$$\mathbb{I}_{T^{-1}A^c}(x) = \mathbb{I}_X(x) - \mathbb{I}_{T^{-1}A}(x) = 1 - \mathbb{I}_A(x) = \mathbb{I}_{A^c}(x).$$
This shows that $\mathbb{I}_{T^{-1}A^c} = \mathbb{I}_{A^c}$ $\mu$ a.e. so that $A^c \in \mathcal{D}$. Finally, let $(A_N)_n$ be a pairwise disjoint sequence in $\mathcal{D}$. For each $n$, let $N_n = \{x \in X : \mathbb{I}_{A_n}(x) \neq \mathbb{I}_{T^{-1}A_n}(x)\}$ and set $N = \bigcup_n N_n$. Then by assumption $\mu(N_n) = 0$ for all $n$ so that $\mu(N) = 0$. For $x \in N^c$, we have
$$\mathbb{I}_{T^{-1}\dot\bigcup_n A_n}(x) = \mathbb{I}_{\dot\bigcup_n T^{-1}A_n}(x) = \sum_n \mathbb{I}_{T^{-1}A_n}(x) = \sum_n \mathbb{I}_{A_n}(x) = \mathbb{I}_{\dot\bigcup_n A_n}(x).$$
Thus, $\mathbb{I}_{T^{-1}\dot\bigcup_n A_n} = \mathbb{I}_{\dot\bigcup_n A_n}$ $\mu$ a.e. and hence $\dot\bigcup_n A_n \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a Dynkin system.

**Proof (b)**: Let $A, B \in \mathcal{D}$ and set $N_1 = \{x \in X : \mathbb{I}_A(x) \neq \mathbb{I}_{T^{-1}A}(x)\}$, $N_2 = \{x \in X : \mathbb{I}_B(x) \neq \mathbb{I}_{T^{-1}B}(x)\}$ and $N = N_1 \cup N_2$. Then $\mu(N_1) = \mu(N_2) = \mu(N) = 0$, and for $x \in N^c$ we have
$$\mathbb{I}_{T^{-1}(A\cap B)}(x) = \mathbb{I}_{T^{-1}A \cap T^{-1}B}(x) = \mathbb{I}_{T^{-1}A}(x)\mathbb{I}_{T^{-1}B}(x) = \mathbb{I}_A(x)\mathbb{I}_B(x) = \mathbb{I}_{A\cap B}(x).$$
Thus, $\mathbb{I}_{T^{-1}(A\cap B)} = \mathbb{I}_{A\cap B}$ $\mu$ a.e. so that $A \cap B \in \mathcal{D}$. Therefore, $\mathcal{D}$ is $\bigcap$-stable.

**Proof (c)**: Parts (a), (b) and Lemma 5.4 imply that $\mathcal{D}$ is a $\sigma$-algebra.

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra, and $\lambda$ Lebesgue measure. Prove that
$$\lim_{n\to\infty} \int_{[0,n]} (1 + \frac{x}{n})^{-n}(1 - \sin\frac{x}{n})\, d\lambda(x) = 1.$$
(Hint: The positive sequence $\left((1+\frac{x}{n})^{-n}\mathbb{I}_{[0,n]}(x)\right)_n$ decreases to $e^{-x}\mathbb{I}_{[0,\infty)}(x)$) (2 pts)

**Proof**: Let $u_n(x) = \mathbb{I}_{[0,n]}(x)(1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})$. The positive sequence $\left((1+\frac{x}{n})^{-n}\mathbb{I}_{[0,n]}(x)\right)_n$ decreases to $e^{-x}\mathbb{I}_{[0,\infty)}(x)$ and the sequence $\left(1-\sin\frac{x}{n}\right)_n$ is bounded from below by 0 and from above by 2 and converges to 1 as $n \to \infty$. Thus, $\lim_{n\to\infty} u_n(x) = \mathbb{I}_{[0,\infty)}(x)e^{-x}$, and $0 \le u_n(x) \le 2(1+\frac{x}{2})^{-2}\mathbb{I}_{[0,\infty)}(x)$ for $n \ge 2$ and all $x \in \mathbb{R}$. Since the function $2(1+\frac{x}{2})^{-2}\mathbb{I}_{[0,\infty)}(x)$ is measurable, non-negative and the improper Riemann integrable on $[0,\infty)$ exists, it follows that it is Lebesgue integrable on $[0,\infty)$. By Lebesgue Dominated Convergence Theorem (and taking the limit for $n \ge 2$), we have
$$\begin{aligned}\lim_{n\to\infty} \int_{[0,n]} (1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\, d\lambda(x) &= \lim_{n\to\infty} \int u_n(x)d\lambda(x) \\ &= \int \mathbb{I}_{[0,\infty)}(x)e^{-x}\, d\lambda(x) = \int_0^\infty e^{-x}\, dx = 1.\end{aligned}$$

(3) Consider the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda \times \lambda)$, where $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ is the Borel product $\sigma$-algebra and $\lambda \times \lambda$ is Lebesgue product measure. For $E \in \mathcal{B}(\mathbb{R}^2)$ and $x, y \in \mathbb{R}$, set $E_x = \{y \in \mathbb{R} : (x,y) \in E\}$ and $E_y = \{x \in \mathbb{R} : (x,y) \in E\}$.

(a) Let $E \in \mathcal{B}(\mathbb{R}^2)$ and assume $\lambda(E_x) = 0$ for $\lambda$ a.e. $x$. Prove that $(\lambda \times \lambda)(E) = 0$ and that $\lambda(E_y) = 0$ for $\lambda$ a.e. $y$. (1.5 pts)

(b) Let $f \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{B}(\mathbb{R}^2))$. Set $F = \{(x,y) \in \mathbb{R}^2 : f(x,y) = \infty\}$ and assume $\lambda(F_y) = 0$ for $\lambda$ a.e. $y$. Prove that $F \in \mathcal{B}_{\overline{\mathbb{R}}}(\mathbb{R}^2)$ and that $(\lambda \times \lambda)(F) = 0$. (0.5 pt)

**Proof (a)**: By Theorem 14.5, the sets $E_x, E_y \in \mathcal{B}(\mathbb{R})$ and

$$(\lambda \times \lambda)(E) = \int_{\mathbb{R}} \lambda(E_y)\, d\lambda(y) = \int_{\mathbb{R}} \lambda(E_x)\, d\lambda(x),$$

note that $\mathbb{I}_E(x,y) = \mathbb{I}_{E_x}(y) = \mathbb{I}_{E_y}(x)$. By assumption $\lambda(E_x) = 0$ for $\lambda$ a.e. $x$, hence by Corollary 11.3 (ii)

$$(\lambda \times \lambda)(E) = \int_{\mathbb{R}} \lambda(E_y)\, d\lambda(y) = 0.$$

This shows that $(\lambda \times \lambda)(E) = 0$ and by Theorem 11.2 (i) we have $\lambda(E_y) = 0$ for $\lambda$ a.e. $y$.

**Proof (b)**: Note that $\{\infty\} \in \mathcal{B}(\overline{\mathbb{R}})$, and since $f$ is measurable we have $F = f^{-1}(\{\infty\}) \in \mathcal{B}_{\overline{\mathbb{R}}}(\mathbb{R}^2)$. Using the fact that $\lambda(F_y) = 0$ for $\lambda$ a.e. $y$ and part (a) (interchanging the roles of $x$ and $y$), we see that $(\lambda \times \lambda)(F) = 0$.

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1(\mu) \bigcap \mathcal{L}^\infty(\mu)$. Set $E = \{x \in X : |f(x)| \geq 1\}$.

(a) Prove that $\mu(E) < \infty$ and that for any $p \in [1, \infty)$ one has $\left(\int_E |f|^p\, d\mu\right)^{1/p} \leq \|f\|_\infty \mu(E)^{1/p}$. (1.5 pts)

(b) Prove that $f \in \mathcal{L}^p(\mu)$ for all $p \in [1, \infty)$. (1.5 pts)

**Proof (a)**: By hypothesis, we have $\int_X |f|\, d\mu < \infty$ and $|f| \leq \|f\|_\infty < \infty$ $\mu$ a.e. Thus,

$$\mu(E) = \int_E 1\, d\mu \leq \int_E |f|\, d\mu \leq \int_X |f|\, d\mu < \infty.$$

Let $p \in [1, \infty)$, using $|f| \leq \|f\|_\infty < \infty$ $\mu$ a.e. we have

$$\int_E |f|^p\, d\mu \leq \|f\|_\infty^p \int_E 1\, d\mu = \|f\|_\infty^p \mu(E).$$

Thus, $\left(\int_E |f|^p\, d\mu\right)^{1/p} \leq \|f\|_\infty \mu(E)^{1/p}$.

**Proof (b)**: Note that for $x \in E^c$, one has $|f(x)| < 1$ so that $|f(x)|^p \leq |f(x)|$. Using this and part (a), we have

$$\begin{aligned}
\int_X |f|^p\, d\mu &= \int_E |f|^p\, d\mu + \int_{E^c} |f|^p\, d\mu \\
&\leq \|f\|_\infty^p \mu(E) + \int_{E^c} |f|\, d\mu \\
&\leq \|f\|_\infty^p \mu(E) + \int_X |f|\, d\mu \\
&< \infty.
\end{aligned}$$

Thus, $f \in \mathcal{L}^p(\mu)$ for all $p \in [1, \infty)$.