Measure and Integration: Retake Exam 2021-22

(1) Let $(X, \mathcal{A}, \mu)$ be a measure space and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$-measurable transformation. Let
$$\mathcal{D} = \{A \in \mathcal{A} : \mathbb{I}_A = \mathbb{I}_{T^{-1}A} \ \ \mu \text{ a.e.}\}.$$

(a) Prove that $\mathcal{D}$ is a Dynkin system. (1.5 pts)

(b) Prove that $\mathcal{D}$ is $\bigcap$-stable. (1 pt)

(c) Prove that $\mathcal{D}$ is a σ-algebra. (0.5 pts)

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Prove that
$$\lim_{n\to\infty} \int_{[0,n]} (1 + \frac{x}{n})^{-n} (1 - \sin \frac{x}{n})\, d\lambda(x) = 1.$$
(Hint: The positive sequence $\left((1 + \frac{x}{n})^{-n} \mathbb{I}_{[0,n]}(x)\right)_n$ decreases to $e^{-x}\mathbb{I}_{[0,\infty)}(x)$) (2 pts)

(3) Consider the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda \times \lambda)$, where $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ is the Borel product σ-algebra and $\lambda \times \lambda$ is Lebesgue product measure. For $E \in \mathcal{B}(\mathbb{R}^2)$ and $x, y \in \mathbb{R}$, set $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ and $E_y = \{x \in \mathbb{R} : (x, y) \in E\}$.

(a) Let $E \in \mathcal{B}(\mathbb{R}^2)$ and assume $\lambda(E_x) = 0$ for λ a.e. x. Prove that $(\lambda \times \lambda)(E) = 0$ and that $\lambda(E_y) = 0$ for λ a.e. y. (1.5 pts)

(b) Let $f \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{B}(\mathbb{R}^2))$. Set $F = \{(x, y) \in \mathbb{R}^2 : f(x, y) = \infty\}$ and assume $\lambda(F_y) = 0$ for λ a.e. y. Prove that $F \in \mathcal{B}_{\overline{\mathbb{R}}}(\mathbb{R}^2)$ and that $(\lambda \times \lambda)(F) = 0$. (0.5 pt)

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1(\mu) \bigcap \mathcal{L}^\infty(\mu)$. Set $E = \{x \in X : |f(x)| \geq 1\}$.

(a) Prove that $\mu(E) < \infty$ and that for any $p \in [1, \infty)$ one has $\left(\int_E |f|^p\, d\mu\right)^{1/p} \leq \|f\|_\infty \mu(E)^{1/p}$. (1.5 pts)

(b) Prove that $f \in \mathcal{L}^p(\mu)$ for all $p \in [1, \infty)$. (1.5 pts)