Measure and Integration: Solutions Mid-Term, 2021-22

(1) Let $X = (0,1]$ and $\mathcal{G} = \{(a,b]: 0 \leq a < b \leq 1\} \cup \{\emptyset\}$. Consider the collection $\mathcal{F}$ consisting of all subsets of $X$ that can be written as a finite disjoint union of elements of $\mathcal{G}$.

(a) Prove that if $A \in \mathcal{F}$ then $A^c = X \setminus A \in \mathcal{F}$. (1 pt)

(b) Prove that if $A,B \in \mathcal{F}$, then $A \cap B, A \cup B, A \setminus B \in \mathcal{F}$. (2 pts)

(c) Prove that $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$. (1 pt)

**Proof (a):** Let $A \in \mathcal{F}$, then $A = \bigcup_{i=1}^{n} (a_i, b_i]$ for some $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq 1$. We consider four cases, and we use the convention that if $b_i = a_{i+1}$, then $(b_i, a_{i+1}] = \emptyset \in \mathcal{G}$. To this end, if $a_1 = 0, b_n = 1$, then $A^c = \bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \in \mathcal{F}$. If $a_1 = 0, b_n < 1$, then $A^c = \bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \cup (b_n, 1] \in \mathcal{F}$. If $a_1 > 0, b_n = 1$, then $A^c = (0, a_1] \cup \bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \in \mathcal{F}$. If $a_1 > 0, b_n < 1$, then $A^c = (0, a_1] \cup \bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \cup (b_n, 1] \in \mathcal{F}$.

**Proof (b):** Let $A, B \in \mathcal{F}$, then $A = \bigcup_{i=1}^{n} (a_i, b_i]$ for some $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq 1$ and $B = \bigcup_{j=1}^{m} (c_j, d_j]$ for some $0 \leq c_1 < d_1 \leq c_2 < d_2 \leq \cdots \leq c_m < d_m \leq 1$. First note that $\mathcal{G}$ is $\cap$-stable since if $(a,b], (c,d] \in \mathcal{G}$, then $(a,b] \cap (c,d] = (\max(a,c), \min(b,d)] \in \mathcal{G}$ (if $\min(b,d) \leq \max(a,c)$ then this interval is empty, hence an element of $\mathcal{G}$). Thus,

$$A \cap B = \bigcup_{j=1}^{m} \bigcap_{i=1}^{n} ((a_i, b_i] \cap (c_j, d_j]) \in \mathcal{F}.$$  

From this and part (a), we have $A \cup B = (A^c \cap B^c)^c \in \mathcal{F}$ and $A \setminus B = A \cap B^c \in \mathcal{F}$.

**Proof (c):** Clearly $\mathcal{G} \subset \mathcal{F}$, hence $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F})$. Now, every element of $\mathcal{F}$ is either the empty set, which is an element of $\sigma(\mathcal{G})$, or is non-empty and hence a finite disjoint union of elements of $\mathcal{G}$ and therefore a member of $\sigma(\mathcal{G})$. This shows that $\mathcal{F} \subset \sigma(\mathcal{G})$ so that $\sigma(\mathcal{F}) \subset \sigma(\mathcal{G})$. Combining both containments leads to $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$.

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra and $\lambda$ is Lebesgue measure. Let $E \in \mathcal{B}(\mathbb{R})$ with $\lambda(E) < \infty$, and define $\varphi_E : \mathbb{R} \rightarrow [0, \infty)$ by $\varphi_E(x) = \lambda(E \cap (-\infty, x))$.

(a) Prove that $\varphi_E$ is an increasing function. (0.5 pts)

(b) Prove that $\lim_{x \to -\infty} \varphi_E(x) = \lambda(E)$ and $\lim_{x \to +\infty} \varphi_E(x) = 0$. (1.5 pts)

(c) Prove that for any $x, x' \in \mathbb{R}$, one has

$$|\varphi_E(x) - \varphi_E(x')| \leq |x - x'|.$$  

Conclude that $\varphi_E$ is uniformly continuous. (1.5 pts)

**Proof (a):** Let $x \leq x'$, then $(-\infty, x] \subseteq (-\infty, x']$ and hence $E \cap (-\infty, x) \subseteq E \cap (-\infty, x')$. By monotonicity of $\lambda$, we have

$$\varphi_E(x) = \lambda\left(E \cap (-\infty, x]\right) \leq \lambda\left(E \cap (-\infty, x')\right) = \varphi_E(x').$$

This shows that $\varphi_E$ is an increasing function.
Proof (b): By part (a), to prove \( \lim_{x \to +\infty} \varphi_E(x) = \lambda(E) \) it is enough to show that for any increasing sequence \( x_n \nearrow +\infty \), one has \( \lim_{n \to +\infty} \varphi_E(x_n) = \lambda(E) \). So let \( x_n \nearrow +\infty \), then \( E \cap (-\infty, x_n) \nearrow E \). So by continuity from below (Proposition 4.3 (vi)) one has
\[
\lim_{n \to +\infty} \varphi_E(x_n) = \lim_{n \to +\infty} \lambda\left(E \cap (-\infty, x_n)\right) = \lambda(E).
\]
Similarly, to show \( \lim_{x \to -\infty} \varphi_E(x) = 0 \), it is enough to show for any decreasing sequence \( y_n \searrow -\infty \) one has \( \lim_{n \to +\infty} \varphi_E(y_n) = 0 \). So let \( y_n \searrow -\infty \), then \( E \cap (-\infty, y_n) \searrow \emptyset \). Since \( \lambda\left(E \cap (-\infty, y_1)\right) < \infty \), by Proposition 4.3 (vii) we have
\[
\lim_{n \to +\infty} \varphi_E(y_n) = \lim_{n \to +\infty} \lambda\left(E \cap (-\infty, y_n)\right) = \lambda(\emptyset) = 0.
\]
Proof (c): Let \( x, x' \in \mathbb{R} \) and assume with no loss of generality that \( x \leq x' \). Since \( \varphi_E \) is increasing, we have
\[
\left| \varphi_E(x) - \varphi_E(x') \right| = \varphi_E(x') - \varphi_E(x)
= \lambda\left(E \cap (-\infty, x')\right) - \lambda\left(E \cap (-\infty, x)\right)
= \lambda\left(E \cap (-\infty, x') \setminus E \cap (-\infty, x)\right)
= \lambda\left(E \cap [x, x')\right)
\leq \lambda([x, x')]
= x' - x
= |x - x'|,
\]
where the third equality follows from Proposition 4.3 (iii), since \( (-\infty, x) \subseteq (-\infty, x') \) and \( \lambda\left(E \cap (-\infty, x)\right) \leq \lambda(E) < \infty \). Finally, let \( \epsilon > 0 \) and choose \( \delta = \epsilon \). Then from the above, we have for any \( x, x' \) with \( |x - x'| < \delta \),
\[
\left| \varphi_E(x) - \varphi_E(x') \right| \leq |x - x'| < \delta = \epsilon.
\]
This proves that \( \varphi_E \) is absolutely continuous.

(3) Consider the measure space \( ([0, 1], \mathcal{B}([0, 1]), \lambda) \), where \( \mathcal{B}([0, 1]) \) is the Borel \( \sigma \)-algebra restricted to \( [0, 1] \) and \( \lambda \) is the restriction of Lebesgue measure on \( [0, 1] \). Define a map \( T: [0, 1) \to [0, 1) \) by
\[
T(x) = \sum_{n=1}^{\infty} \left( n(n+1)x - n \right) \cdot I_{\left(\frac{1}{n^2+1}, \frac{1}{n^2}\right)}(x),
\]
where \( I_A \) denotes the indicator function of the set \( A \).

(a) Show that \( T \) is \( \mathcal{B}([0, 1))/\mathcal{B}([0, 1]) \) measurable. (1 pt)

(b) Determine the image measure \( T(\lambda) = \lambda \circ T^{-1} \) and prove that \( T(\lambda) = \lambda \). (Hint: \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \) ) (1.5 pts)

Proof(a): To show \( T \) is \( \mathcal{B}([0, 1))/\mathcal{B}([0, 1]) \) measurable, by Lemma 7.2 it is enough to consider pre-images of intervals of the form \([a, b] \subseteq [0, 1]\). Now,
\[
T^{-1}([a, b]) = \begin{cases} 
\{0\} \cup \bigcup_{n=1}^{\infty} \left[ \frac{a + n}{n(n+1)}, \frac{b + n}{n(n+1)} \right] & a = 0, \\
\bigcup_{n=1}^{\infty} \left[ \frac{a + n}{n(n+1)}, \frac{b + n}{n(n+1)} \right] & a > 0.
\end{cases}
\]
Since \( \{0\} \in \mathcal{B}([0, 1]) \), we see that in both cases \( T^{-1}([a, b]) \in \mathcal{B}([0, 1]) \) so \( T \) is \( \mathcal{B}([0, 1))/\mathcal{B}([0, 1]) \) measurable.
Another quick way of showing measurability of $T$ is to notice that the functions $x \to n(n + 1)x - n$ are continuous and hence Borel measurable. Also $I_{\left(\frac{1}{n+1},\frac{1}{n}\right)}$ is Borel measurable since $\left[\frac{1}{n+1},\frac{1}{n}\right] \in \mathcal{B}([0,1))$. By Corollary 8.11, the product $x \to (n(n + 1)x - n) \cdot I_{\left(\frac{1}{n+1},\frac{1}{n}\right)}(x)$ is Borel measurable as well as the finite sum $\sum_{j=1}^{n} (j(j+1)x - j) \cdot I_{\left(\frac{1}{j+1},\frac{1}{j}\right)}(x)$ for all $n \geq 1$. Finally, by Corollary 8.10, $T(x) = \lim_{n \to \infty} \sum_{j=1}^{n} (j(j+1)x - j) \cdot I_{\left(\frac{1}{j+1},\frac{1}{j}\right)}(x)$ is Borel measurable.

**Proof (b):** We claim that $T(\lambda) = \lambda$. To prove this, we use Theorem 5.7. Notice that $\mathcal{B}([0,1))$ is generated by the collection $\mathcal{G} = \{[a,b] : 0 \leq a \leq b \leq 1\}$ which is closed under finite intersections. Since $\lambda(\{0\}) = 0$, we have

$$T(\lambda)([a,b]) = \lambda(T^{-1}([a,b]))$$

$$= \lambda\left(\bigcup_{n=1}^{\infty} [a + \frac{n}{n(n+1)}, b + \frac{n}{n(n+1)}]\right)$$

$$= \sum_{n=1}^{\infty} \frac{b-a}{n(n+1)}$$

$$= b-a$$

$$= \lambda([a,b]).$$

Since the constant sequence $([0,1))$ is exhausting, belongs to $\mathcal{G}$ and $\lambda([0,1)) = T(\lambda([0,1))) = 1 < \infty$, we have by Theorem 5.7 that $T(\lambda) = \lambda$. 