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Measure and Integration: Solutions Mid-Term, 2021-22

(1) Let $X = (0,1]$ and $\mathcal{G} = \{(a,b] : 0 \le a < b \le 1\} \cup \{\varnothing\}$. Consider the collection $\mathcal{F}$ consisting of all subsets of X that can be written as a finite disjoint union of elements of $\mathcal{G}$.

(a) Prove that if $A \in \mathcal{F}$ then $A^c = X \setminus A \in \mathcal{F}$. (1 pt)

(b) Prove that if $A, B \in \mathcal{F}$, then $A \cap B, A \cup B, A \setminus B \in \mathcal{F}$. (2 pts)

(c) Prove that $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$. (1 pt)

Proof (a): Let $A \in \mathcal{F}$, then $A = \dot\bigcup_{i=1}^n (a_i, b_i]$ for some $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n \le 1$. We consider four cases, and we use the convention that if $b_i = a_{i+1}$, then $(b_i, a_{i+1}] = \varnothing \in \mathcal{G}$. To this end, if $a_1 = 0, b_n = 1$, then $A^c = \dot\bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \in \mathcal{F}$. If $a_1 = 0, b_n < 1$, then $A^c = \dot\bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \dot\bigcup (b_n, 1] \in \mathcal{F}$. If $a_1 > 0, b_n = 1$, then $A^c = (0, a_1] \dot\cup \dot\bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \in \mathcal{F}$. If $a_1 > 0, b_n < 1$, then $A^c = (0, a_1] \dot\cup \dot\bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \dot\bigcup (b_n, 1] \in \mathcal{F}$.

Proof (b): Let $A, B \in \mathcal{F}$, then $A = \dot\bigcup_{i=1}^n (a_i, b_i]$ for some $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n \le 1$ and $B = \dot\bigcup_{i=1}^n (c_i, d_i]$ for some $0 \le c_1 < d_1 \le c_2 < d_2 \le \cdots \le c_n < d_n \le 1$. First note that $\mathcal{G}$ is $\bigcap$-stable since if $(a,b], (c,d] \in \mathcal{G}$, then $(a,b] \cap (c,d] = \big(\max(a,c), \min(b,d)\big] \in \mathcal{G}$ (if $\min(b,d) \le \max(a,c)$ then this interval is empty, hence an element of $\mathcal{G}$). Thus,

$$A \cap B = \dot\bigcup_{j=1}^m \dot\bigcup_{i=1}^n \big((a_i, b_i] \cap (c_j, d_j]\big) \in \mathcal{F}.$$

From this and part (a), we have $A \cup B = (A^c \cap B^c)^c \in \mathcal{F}$ and $A \setminus B = A \cap B^c \in \mathcal{F}$.

Proof (c): Clearly $\mathcal{G} \subset \mathcal{F}$, hence $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$. Now, every element of $\mathcal{F}$ is either the empty set, which is an element of $\sigma(\mathcal{G})$, or is non-empty and hence a finite disjoint union of elements of $\mathcal{G}$ and therefore a member of $\sigma(\mathcal{G})$. This shows that $\mathcal{F} \subset \sigma(\mathcal{G})$ so that $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G})$. Combining both containments leads to $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$.

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra and λ is Lebesgue measure. Let $E \in \mathcal{B}(\mathbb{R})$ with $\lambda(E) < \infty$, and define $\varphi_E : \mathbb{R} \to [0, \infty)$ by $\varphi_E(x) = \lambda\big(E \cap (-\infty, x)\big)$.

(a) Prove that φ_E is an increasing function. (0.5 pts)

(b) Prove that $\lim_{x \to +\infty} \varphi_E(x) = \lambda(E)$ and $\lim_{x \to -\infty} \varphi_E(x) = 0$. (1.5 pts)

(c) Prove that for any $x, x' \in \mathbb{R}$, one has

$$\Big|\varphi_E(x) - \varphi_E(x')\Big| \le |x - x'|.$$

Conclude that φ_E is uniformly continuous. (1.5 pts)

Proof (a): Let $x \le x'$, then $(-\infty, x) \subseteq (-\infty, x')$ and hence $E \cap (-\infty, x) \subseteq E \cap (-\infty, x')$. By monotonicity of λ, we have

$$\varphi_E(x) = \lambda\Big(E \cap (-\infty, x)\Big) \le \lambda\Big(E \cap (-\infty, x')\Big) = \varphi_E(x).$$

This shows that φ_E is an increasing function.

Proof (b): By part (a), to prove $\lim_{x\to+\infty} \varphi_E(x) = \lambda(E)$ it is enough to show that for any increasing sequence $x_n \nearrow +\infty$, one has $\lim_{n\to+\infty} \varphi_E(x_n) = \lambda(E)$. So let $x_n \nearrow +\infty$, then $E \cap (-\infty, x_n) \nearrow E$. So by continuity from below (Proposition 4.3 (vi)) one has

$$\lim_{n\to+\infty} \varphi_E(x_n) = \lim_{n\to+\infty} \lambda\Big(E \cap (-\infty, x_n)\Big) = \lambda(E).$$

Similarly, to show $\lim_{x\to-\infty} \varphi_E(x) = 0$, it is enough to show for any decreasing sequence $y_n \searrow -\infty$ one has $\lim_{n\to+\infty} \varphi_E(y_n) = 0$. So let $y_n \searrow -\infty$, then $E \cap (-\infty, y_n) \searrow \varnothing$. Since $\lambda\Big(E \cap (-\infty, y_1)\Big) < \infty$, by Proposition 4.3 (vii) we have

$$\lim_{n\to+\infty} \varphi_E(y_n) = \lim_{n\to+\infty} \lambda\Big(E \cap (-\infty, y_n)\Big) = \lambda(\varnothing) = 0.$$

Proof (c): Let $x, x' \in \mathbb{R}$ and assume with no loss of generality that $x \le x'$. Since φ_E is increasing, we have

$$\begin{aligned}
\Big|\varphi_E(x) - \varphi_E(x'\Big| &= \varphi_E(x') - \varphi_E(x) \\
&= \lambda\Big(E \cap (-\infty, x')\Big) - \lambda\Big(E \cap (-\infty, x)\Big) \\
&= \lambda\Big(E \cap (-\infty, x') \setminus E \cap (-\infty, x)\Big) \\
&= \lambda\Big(E \cap [x, x')\Big) \\
&\le \lambda\Big([x, x')\Big) \\
&= x' - x \\
&= |x - x'|,
\end{aligned}$$

where the third equality follows from Proposition 4.3 (iii), since $(-\infty, x) \subseteq (-\infty, x')$ and $\lambda\Big(E \cap (-\infty, x)\Big) \le \lambda(E) < \infty$. Finally, let $\epsilon > 0$ and choose $\delta = \epsilon$. Then from the above, we have for any x, x' with $|x - x'| < \delta$,

$$\Big|\varphi_E(x) - \varphi_E(x'\Big| \le |x - x'| < \delta = \epsilon.$$

This proves that φ_E is absolutely continuous.

(3) Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define a map $T : [0,1) \to [0,1)$ by

$$T(x) = \sum_{n=1}^{\infty} \Big(n(n+1)x - n\Big) \cdot \mathbb{I}_{[\frac{1}{n+1}, \frac{1}{n})}(x),$$

where $\mathbb{I}_A$ denotes the indicator function of the set A.

(a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable. (1 pt)

(b) Determine the image measure $T(\lambda) = \lambda \circ T^{-1}$ and prove that $T(\lambda) = \lambda$. (Hint: $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$) (1.5 pts)

Proof(a): To show T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable, by Lemma 7.2 it is enough to consider pre-mages of intervals of the form $[a,b) \subset [0,1)$. Now,

$$T^{-1}([a,b)) = \begin{cases} \{0\} \cup \dot{\bigcup}_{n=1}^{\infty} \Big[\dfrac{a+n}{n(n+1)}, \dfrac{b+n}{n(n+1)}\Big) & a = 0, \\[2ex] \dot{\bigcup}_{n=1}^{\infty} \Big[\dfrac{a+n}{n(n+1)}, \dfrac{b+n}{n(n+1)}\Big), & a > 0. \end{cases}$$

Since $\{0\} \in \mathcal{B}([0,1))$, we see that in both cases $T^{-1}([a,b)) \in \mathcal{B}([0,1))$ so T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable.

Another quick way of showing measurability of T is to notice that the functions $x \to n(n+1)x - n$ are continuous and hence Borel measurable. Also $\mathbb{I}_{[\frac{1}{n+1}, \frac{1}{n})}$ is Borel measurable since $[\frac{1}{n+1}, \frac{1}{n}) \in \mathcal{B}([0,1))$. By Corollary 8.11, the product $x \to \big(n(n+1)x - n\big) \cdot \mathbb{I}_{[\frac{1}{n+1}, \frac{1}{n})}(x)$ is Borel measurable as well as the finite sum $\sum_{j=1}^{n} \big(j(j+1)x - j\big) \cdot \mathbb{I}_{[\frac{1}{j+1}, \frac{1}{j})}(x)$ for all $n \geq 1$. Finally, by Corollary 8.10, $T(x) = \lim_{n\to\infty} \sum_{j=1}^{n} \big(j(j+1)x - j\big) \cdot \mathbb{I}_{[\frac{1}{j+1}, \frac{1}{j})}(x)$ is Borel measurable.

Proof(b): We claim that $T(\lambda) = \lambda$. To prove this, we use Theorem 5.7. Notice that $\mathcal{B}([0,1))$ is generated by the collection $\mathcal{G} = \{[a,b) : 0 \leq a \leq b \leq 1\}$ which is closed under finite intersections. Since $\lambda(\{0\}) = 0$, we have

$$\begin{aligned} T(\lambda)([a,b)) &= \lambda(T^{-1}([a,b))) \\ &= \lambda\Big(\dot{\bigcup_{n=1}^{\infty}} \Big[\frac{a+n}{n(n+1)}, \frac{b+n}{n(n+1)}\Big)\Big) \\ &= \sum_{n=1}^{\infty} \frac{b-a}{n(n+1)} \\ &= b-a \\ &= \lambda([a,b)). \end{aligned}$$

Since the constant sequence $([0,1))$ is exhausting, belongs to $\mathcal{G}$ and $\lambda([0,1)) = T(\lambda([0,1)) = 1 < \infty$, we have by Theorem 5.7 that $T(\lambda) = \lambda$.