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Measure and Integration: Mid-Term, 2021-22

(1) Let $X = (0,1]$ and $\mathcal{G} = \{(a,b] : 0 \le a < b \le 1\} \cup \{\varnothing\}$. Consider the collection $\mathcal{F}$ consisting of all subsets of X that can be written as a finite disjoint union of elements of $\mathcal{G}$.

(a) Prove that if $A \in \mathcal{F}$ then $A^c = X \setminus A \in \mathcal{F}$. (1 pt)

(b) Prove that if $A, B \in \mathcal{F}$, then $A \cap B, A \cup B, A \setminus B \in \mathcal{F}$. (2 pts)

(c) Prove that $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$. (1 pt)

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra and λ is Lebesgue measure. Let $E \in \mathcal{B}(\mathbb{R})$ with $\lambda(E) < \infty$, and define $\varphi_E : \mathbb{R} \to [0,\infty)$ by $\varphi_E(x) = \lambda\big(E \cap (-\infty, x)\big)$.

(a) Prove that φ_E is an increasing function. (0.5 pts)

(b) Prove that $\lim\limits_{x\to+\infty} \varphi_E(x) = \lambda(E)$ and $\lim\limits_{x\to-\infty} \varphi_E(x) = 0$. (1.5 pts)

(c) Prove that for any $x, x' \in \mathbb{R}$, one has

$$\Big|\varphi_E(x) - \varphi_E(x')\Big| \le |x - x'|.$$

Conclude that φ_E is uniformly continuous. (1.5 pts)

(3) Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define a map $T : [0,1) \to [0,1)$ by

$$T(x) = \sum_{n=1}^{\infty} \big(n(n+1)x - n\big) \cdot \mathbb{I}_{[\frac{1}{n+1}, \frac{1}{n})}(x),$$

where $\mathbb{I}_A$ denotes the indicator function of the set A.

(a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable. (1 pt)

(b) Determine the image measure $T(\lambda) = \lambda \circ T^{-1}$ and prove that $T(\lambda) = \lambda$. (Hint: $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$) (1.5 pts)