Measure and Integration: Mid-Term, 2021-22

(1) Let \( X = (0, 1) \) and \( \mathcal{G} = \{(a, b) : 0 \leq a < b \leq 1\} \cup \{\emptyset\} \). Consider the collection \( \mathcal{F} \) consisting of all subsets of \( X \) that can be written as a finite disjoint union of elements of \( \mathcal{G} \).

(a) Prove that if \( A \in \mathcal{F} \) then \( A^c = X \setminus A \in \mathcal{F} \). (1 pt)

(b) Prove that if \( A, B \in \mathcal{F} \), then \( A \cap B, A \cup B, A \setminus B \in \mathcal{F} \). (2 pts)

(c) Prove that \( \sigma(\mathcal{G}) = \sigma(\mathcal{F}) \). (1 pt)

(2) Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\), where \( \mathcal{B}(\mathbb{R}) \) is the Borel \( \sigma \)-algebra and \( \lambda \) is Lebesgue measure. Let \( E \in \mathcal{B}(\mathbb{R}) \) with \( \lambda(E) < \infty \), and define \( \varphi_E : \mathbb{R} \to [0, \infty) \) by \( \varphi_E(x) = \lambda(E \cap (-\infty, x)) \).

(a) Prove that \( \varphi_E \) is an increasing function. (0.5 pts)

(b) Prove that \( \lim_{x \to +\infty} \varphi_E(x) = \lambda(E) \) and \( \lim_{x \to -\infty} \varphi_E(x) = 0 \). (1.5 pts)

(c) Prove that for any \( x, x' \in \mathbb{R} \), one has
\[
\left| \varphi_E(x) - \varphi_E(x') \right| \leq |x - x'|.
\]
Conclude that \( \varphi_E \) is uniformly continuous. (1.5 pts)

(3) Consider the measure space \(([0, 1], \mathcal{B}([0, 1]), \lambda)\), where \( \mathcal{B}([0, 1]) \) is the Borel \( \sigma \)-algebra restricted to \([0, 1]\) and \( \lambda \) is the restriction of Lebesgue measure on \([0, 1]\). Define a map \( T : [0, 1] \to [0, 1] \) by
\[
T(x) = \sum_{n=1}^{\infty} \left( n(n+1)x - n \right) \cdot I_{\left(\frac{n}{n+1}, \frac{1}{n}\right)}(x),
\]
where \( I_A \) denotes the indicator function of the set \( A \).

(a) Show that \( T \) is \( \mathcal{B}([0, 1]) / \mathcal{B}([0, 1]) \) measurable. (1 pt)

(b) Determine the image measure \( T(\lambda) = \lambda \circ T^{-1} \) and prove that \( T(\lambda) = \lambda \). (Hint: \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \)) (1.5 pts)