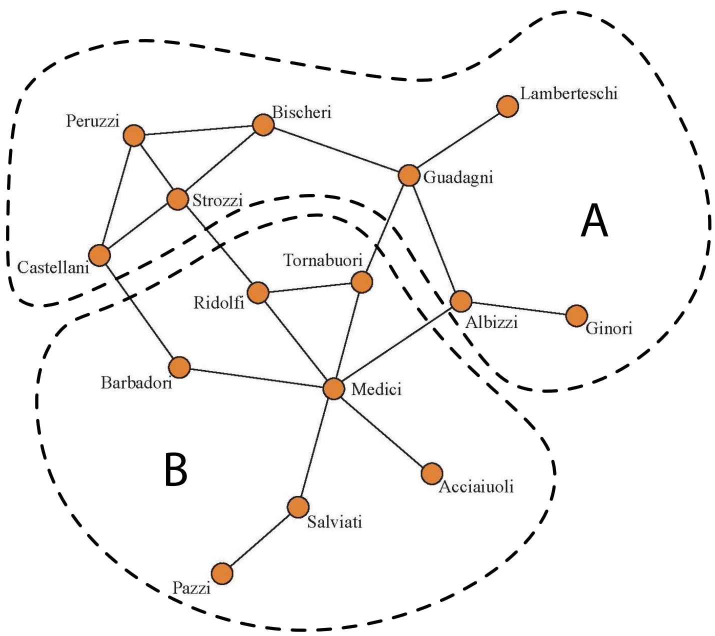
Solution model for: Methods and Models in Complex Systems BETA-B2-CS Final Examenation

November 12, 2021

[MAX 70 points]



1. The figure above, depicts the network of renaissance Florentine families, in which links represent business and marital ties. This network is difficult to cut into two connected components without removing as many as 4 edges, which, one may speculate, had something to do with

relative stability of the network in times of a political turmoil. Nevertheless, being locked in a struggle for political control of the city of Florence in 1430s, two factions eventually appeared dominant in this struggle: one revolved around the powerful Strozzis (A), and the other around the infamous Medicis (B). One may reconstruct these factions, as shown, by computing the sign of the elements of Fiedler's vector shifted to feature 0 median.

(a) *Calculate the empirical degree distribution p_k, that is the fraction of all nodes with degree $k = 1, 2, 3, \ldots$:* $p_1 = \frac{4}{15}$, $p_2 = \frac{2}{15}$, $p_3 = \frac{2}{5}$, $p_4 = \frac{2}{15}$, $p_5 = 0$, $p_6 = \frac{1}{15}$, $p_k = 0$ for $k > 6$.

(b) *Compute the cut quality of the indicated bisection.*
$Q = \frac{\text{links that cross the boundary}}{\text{maximum number of links between A and B}} = \frac{4}{7 \cdot 8} = \frac{1}{14}$.

(c) *Show that the cut quality of the current partition cannot be improved by one family from Strozzis switching sides, that is by moving one node A→B.*
Without switching, $Q = \frac{4}{56}$; with switching it becomes as follows:

Pe	Bi	La	Gu	Al	Gi	Ca	St
7/56	7/56	5/56	6/56	5/56	5/56	5/56	6/56

which are all $> \frac{4}{56}$.

(d) We will now consider a modern network that has ties reaching out to a large number of nodes n. The structure of the network is not relieved, but the degree distribution p_k is known to be the same as in the Florentine network. Suppose one removes edges uniformly at random. Use the random graph formalism, with assumption that $n \to \infty$, to test the network's resilience:

i. *What is the critical fraction of edges π_c that has to be kept to ensure that the network contains the giant component?*

$$\mu_1 = \sum_{k=1}^{6} k p_k = \frac{40}{15} = \frac{8}{3};$$

$$\mu_2 = \sum_{k=1}^{6} k^2 p_k = \frac{134}{15};$$

$$\pi_c = \frac{\mu_1}{\mu_2 - \mu_1} \approx 0.43...$$

ii. *Suppose we now remove all vertices of the largest degree (and renormalise the distribution, ensuring that $\sum_{k\geq 0} p_k = 1$) How does this action affects the critical fraction of edges π_c?*

After we exclude the vertex with degree 6, we have:
$p_1 = \frac{2}{7}$, $p_2 = \frac{1}{7}$, $p_3 = \frac{3}{7}$, $p_4 = \frac{1}{7}$, $p_k = 0$ for $k > 4$.

$$\pi_c = \frac{\mu_1}{\mu_2 - \mu_1} = 17/32 \approx 0.53...$$

Therefore, removing the largest degree node makes the network more fragile.

2. **Evolution of natural languages**
Natural languages may become obsolete. A famous historical example is gradual abundance of Latin, even though speaking this language offered considerable socio-economical benefits. Language extinction is also active today, with 90 percent of world's languages are being expected to disappear by the end of this century.

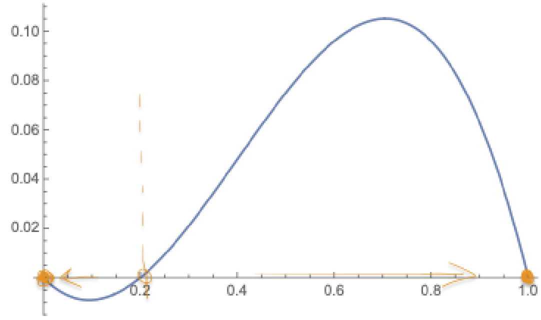
Consider the following model of language competition: let X and Y denote two languages competing for speakers in a given society. The proportion of the population speaking X evolves according to

$$\dot{x} = s(1-x)x^2 - (1-s)x(1-x)^2$$

where $0 \leq x \leq 1$ is the fraction of population speaking X and $1-x$ is the fraction of population speaking Y. Here, $0 \leq s \leq 1$ is the socio-economical advantage of language X over Y.

(a) *Find the fixed points and classify their stability*
There are three fixed points:
$x_1 = 0$ stable for $s < 1$ (unstable for $s = 1$)
$x_2 = 1 - s$ unstable for $s \in (0, 1)$
$x_3 = 1$ stable for $s > 0$ (unstable for $s = 0$)
Furthermore $x_1 < x_2 < x_3$, for $s \in (0, 1)$.

(b) *Draw the phase portrait and argue for which initial conditions a language offering a great socio-economical advantage (s being close to 1), may nevertheless become gradually abundant.*



If s is close to one, but the initial fraction of population using it $x_0 < 1 - s$ is small enough, language X will be eventually be extinct.

3. **Reinfection** With some infections, such as Covid'19, individuals may become temporary immune upon recovery. Hence we distinguish three compartments susceptible (S), infected (I), and immune/recovered (R). Consider the following modification of the SIR model:

$$S + I \xrightarrow{\alpha} 2I$$
$$I \xrightarrow{\beta} R$$
$$R \xrightarrow{\beta} S$$

where $\alpha, \beta > 0$, $\alpha \neq \beta$ are the rates. Let $s(t)$, $x(t)$, and $r(t)$ denote concentration of correspondingly S, I and R species, with $s(t) + x(t) + r(t) = 1$

(a) *Formulate the system of ordinary differential equations for $s(t)$, $x(t)$ and $r(t)$ and show that this system can be well-represented by two differential equation for $s(t), x(t)$, write down these equations.*

$$\begin{cases} \frac{d}{dt}s(t) = \beta r(t) - \alpha s(t)x(t) \\ \frac{d}{dt}x(t) = -\beta x(t) + \alpha s(t)x(t) \\ \frac{d}{dt}r(t) = \beta x(t) - \beta r(t) \\ x(t)+s(t) + r(t) = 1, \end{cases}$$

which can be reduced by using $r(t) = 1 - s(t) - x(t)$:

$$\begin{cases} \frac{d}{dt}s(t) = \beta(1 - s(t) - x(t)) - \alpha s(t)x(t) \\ \frac{d}{dt}x(t) = -\beta x(t) + \alpha s(t)x(t) \end{cases}$$

(b) *Write down the Jacobian matrix for this system of ODEs.*

$$\mathbf{J} = \begin{bmatrix} -\beta - \alpha x & -\beta - \alpha s \\ \alpha x & -\beta + \alpha s \end{bmatrix}$$

(c) *Find all fixed points of the form (s^*, x^*), classify their stability depending on the parameters.*

FP1 (complete recovery):

$$s^* = 1, x^* = 0,\ r^* = 0$$

$$J_1 = \begin{bmatrix} -\beta & -(\alpha + \beta) \\ 0 & \alpha - \beta \end{bmatrix}$$

$$\tau = \alpha - 2\beta,\ det = (\beta - \alpha)\beta$$

FP2 (sustained positive fraction of infected and immune individuals):

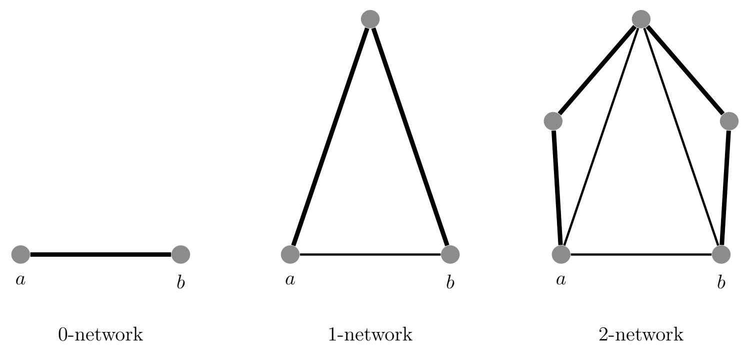
$$s^* = \beta/\alpha,\ x^* = \frac{\alpha - \beta}{2\alpha},\ r^* = \frac{\alpha - \beta}{2\alpha},$$

$$\tau = -\frac{\alpha + \beta}{2},\ det = (\alpha - \beta)\beta$$

Hence: when $\alpha > \beta$ then FP1 is a saddle and FP2 is a stable node; when $\alpha < \beta$ then FP2 is a saddle and FP1 is a stable node.

4. **Hierarchical network**

Consider following iterative construction: A 0-network consist of two initial vertices a and b connected with a link. Iteratively, an $(n+1)$-network is obtained from the n-network by glueing a triangle to each



newly added link at the previous iteration, as shown.

We remove each link with probability $1-p$. We say that a and b are connected with a path, if there is at least one way to travel from a to b by following the links. We are interested in $f_n(p)$, the probability that there is a path from a to b in a fringed n-network. Note that by definition, $f_0(p) = p$, because the only possible path is the link (a, b) itself.

(a) Give a recursive equation for $f_n(p)$.

(b) Show that $f_n(p)$ converges for all $p \in (0,1)$. For which values of p we have that $\lim_{n\to\infty} f_n(p) = 1$?

By definition, we have

$$f_0(p) = p.$$

In 1-network, there are two paths: one may take the initial link, which is present with probability $f_0(p) = p$, or, if it is not present, then there is still a detour of two links, which are simultaneously present with probability p^2. Hence,

$$f_1(p) = p + (1-p)p^2,$$

and, in general, we have a recursion:

$$f_{n+1}(p) = A_p(f_n(p)) := p + (1-p)f_n(p)^2.$$

One can see that $f_n(p)$ is a polynomial. This answers a).

To prove that $f_n(p)$ converges, first note that $f_1(p) > f_0(p)$. Assuming that $f_n(p) > f_{n-1}(p)$ we have

$$f_{n+1}(p) = p + (1-p)f_n(p)^2 > p + (1-p)f_{n-1}(p)^2 = f_n(p).$$

By induction, it follows that $f_n(p)$ is monotonically increasing sequence. Since also $f_n(p) \leq 1$ for all n, it follows that $f_n(p)$ converges.

We will now compute $F(p) := \lim_{n\to\infty} f_n(p)$. Since $A_p(x)$ is continuous, we know that $F(p)$ is a fixed point of A_p. Note that the equation

$$A_p(x) = x$$

has two roots, namely $x_1^* = 1$ and $x_2^* = \frac{p}{1-p}$.

To determine $F(p)$ we study the cases of $p > \frac{1}{2}$ and $p \leq \frac{1}{2}$ separately.

- Let $p > \frac{1}{2}$. Since $f_n(p) \leq 1$ we must have that $F(p) \leq 1$. Because, $p > \frac{1}{2}$, we have $x_2^* = \frac{p}{1-p} > 1$. Therefore, we find that $F(p) = x_1^* = 1$.

- Let $p \leq \frac{1}{2}$. In that case, we have $x_1^* \geq x_2^*$. Since $p < \frac{1}{2}$, we have $f_0(p) = p \leq \frac{p}{1-p}$. Assuming that $f_n(p) \leq \frac{p}{1-p}$, we find that

$$f_{n+1}(p) = p + (1-p)f_n(p)^2 \leq p + (1-p)\frac{p^2}{(1-p)^2} = \frac{p}{1-p}.$$

  Using indunction, we conclude that $f_n(p) \leq \frac{p}{1-p} = x_2^*$ for all n. This implies that $F(p) \leq x_2^*$. Since $x_2^* \leq x_1^*$, it follows that $F(p) = x_2^* = \frac{p}{1-p}$.

Bringing these two cases together, we have

$$F(p) = \begin{cases} \frac{p}{1-p}, & p < \frac{1}{2}, \\ 1, & p \geq \frac{1}{2}. \end{cases}$$

Therefore $\lim_{n\to\infty} f_n(p) = 1$ for $p \geq \frac{1}{2}$.