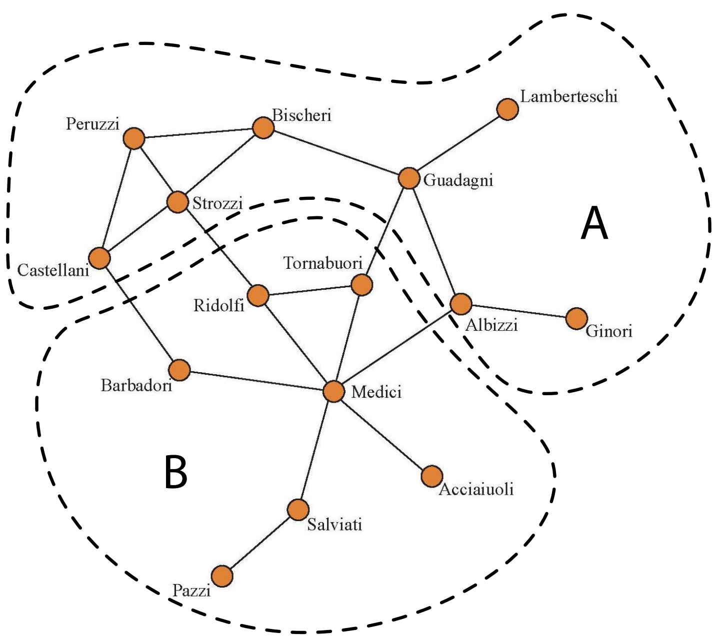
Methods and Models in Complex Systems BETA-B2CSA – Final Examenation

November 12, 2021

[MAX 70 points] For this problems there is no need to use large ($n > 2$) matrix computations or programming. All derivations should be present in your answers. Note, there are 4 problems, and each gives 20 points. If you score more than 70 points, the final score is still capped at 70.



1. **Family ties**

   The figure above, depicts the network of renaissance Florentine families, in which links represent business and marital ties. This network is difficult to cut into two connected components without removing as many as 4 edges, which, one may speculate, had something to do with relative stability of the network in times of a political turmoil. Nevertheless, being locked in a struggle for political control of the city of Florence in 1430s, two factions eventually appeared dominant in this struggle: one revolved around the powerful Strozzis (A), and the other around the infamous Medicis (B). One may reconstruct these factions, as shown, by computing the sign of the elements of Fiedler's vector shifted to feature 0 median, as shown in the figure.

(a) [4 points] Calculate the empirical degree distribution p_k, that is the fraction of nodes with degree $k = 1, 2, 3, \ldots$

(b) [4 points] Compute the cut quality Q of the indicated bisection.

(c) [4 points] Show that the cut quality of the current partition cannot be improved by one family from *Strozzis* switching sides to *Medicis*, that is by moving one node A$\rightarrow$B.

(d) We will now consider a modern network that has ties reaching out to a large number of nodes n. The structure of the network is not relieved, but the degree distribution p_k is known to be the same as in the Florentine network. Suppose one removes edges uniformly at random. Use the random graph formalism, with assumption that $n \to \infty$, to test the network's resilience:

   i. [4 points] What is the critical fraction of edges π_c that has to be kept to ensure that the network contains the giant component?

   ii. [4 points] Suppose we now remove the vertex with the largest degree and renormalise the distribution, ensuring that

$$\sum_{k \geq 0} p_k = 1.$$

   How does this action affects the critical fraction of edges π_c? What does this change mean for the robustness of the network.

2. **Evolution of natural languages**
   Natural languages may become obsolete. A famous historical example is gradual abundance of Latin, even though speaking this language offered considerable socio-economical benefits. Language extinction is also active today, with 90 percent of world's languages are being expected to disappear by the end of this century.

   Consider the following model of language competition: let X and Y denote two languages competing for speakers in a given society. The proportion of the population speaking X evolves according to

$$\dot{x} = s(1-x)x^2 - (1-s)x(1-x)^2$$

   where $0 \leq x \leq 1$ is the fraction of population speaking X and $1-x$ is the fraction of population speaking Y. Here $0 \leq s \leq 1$ is the socio-economical advantage of language X over Y.

   (a) [10 points] Find the fixed points and classify their stability

   (b) [10 points] Draw the phase portrait. Argue for which initial conditions a language offering a great socio-economical advantage (*e.g.* for X, s being close to 1) may nevertheless become gradually abandoned.

3. **Reinfection** With some infections, such as Covid'19, individuals may become temporary immune upon recovery. Therefore, we distinguish three compartments susceptible (S), infected (I), and immune/recovered (R). Consider the following modification of the SIR model:

$$S + I \xrightarrow{\alpha} 2I$$

$$I \xrightarrow{\beta} R$$

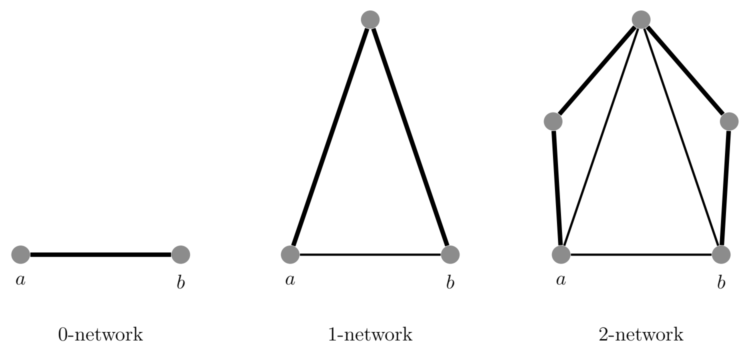
$$R \xrightarrow{\beta} S$$

   where $\alpha, \beta > 0$, $\alpha \neq \beta$ are the rates. Let $s(t)$, $x(t)$, and $r(t)$ denote concentration of correspondingly S, I and R species, with $s(t) + x(t) + r(t) = 1$

   (a) [5 points] Formulate the system of ordinary differential equations for $s(t)$, $x(t)$ and $r(t)$ and show that this system can be well-represented by two differential equation for $s(t), x(t)$, write down these equations.

(b) [5 points] Write down the Jacobian matrix for this system of ODEs.

(c) [10 points] Find all fixed points of the form (s^*, x^*), classify their stability depending on the parameters. Discuss what is the implication of this findings for the reinfection phenomenon.



4. **Hierarchical network**

Consider following iterative construction: A 0-network consist of two initial vertices a and b connected with a link. Iteratively, an $(n+1)$-network is obtained from the n-network by glueing a triangle to each newly added link at the previous iteration, as shown.

We remove each link with probability $1-p$. We say that a and b are connected with a path, if there is at least one way to travel from a to b by following the links. We are interested in $f_n(p)$, the probability that there is a path from a to b in a fringed n-network. Note that by definition, $f_0(p) = p$, because the only possible path is the link (a, b) itself.

(a) [10 points] Give a recursive equation for $f_n(p)$.

(b) [10 points*] Show that $f_n(p)$ converges for all $p \in (0, 1)$. For which values of p, $\lim_{n\to\infty} f_n(p) = 1$?