Vraag 1

$$y = \frac{3}{x} + \frac{1}{5}$$

max.

10 pt.

a) $$fl(y) = \left[\frac{3}{x}(1+\varepsilon_1) + \frac{1}{5}(1+\varepsilon_2)\right](1+\varepsilon_3)$$

$$= \frac{3}{x}(1+\theta_2) + \frac{1}{5}(1+\theta_2') \quad \text{met} \begin{cases} |\theta_2| \leq \frac{2\eta}{1-2\eta} \\ |\theta_2'| \leq \frac{2\eta}{1-2\eta} \end{cases}$$

$$\Rightarrow |fl(y) - y| \leq \frac{2\eta}{1-2\eta}\left(\frac{3}{|x|} + \frac{1}{5}\right)$$

en $$\frac{|fl(y)-y|}{|y|} \leq \frac{2\eta}{1-2\eta} \frac{\frac{3}{|x|} + \frac{1}{5}}{\left|\frac{3}{x} + \frac{1}{5}\right|}$$

max.

8 pt.

$$= \frac{2\eta}{1-2\eta} \frac{3 + \frac{1}{5}|x|}{|3 + \frac{1}{5}x|}$$

$$= \frac{2\eta}{1-2\eta} \frac{15 + |x|}{|15 + x|}$$

b) $x \approx -15$

$x \approx 0$

max.

2pt.

max. 15pt.

Vraag 2

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \gamma > 0$$

a) $\vec{x} = \vec{g}(\vec{x}) \iff \vec{x} = (I - \gamma A)\vec{x} + \gamma \vec{b}$

$\iff \vec{x} = A^{-1}\vec{b}$

$= \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/8 \\ 1/8 \end{pmatrix} = \vec{x}_*$

max. 5pt.

b) $\rho(I - \gamma A)$

$\lambda(I - \gamma A)$: $\begin{vmatrix} 1-3\gamma-\lambda & -\gamma \\ -\gamma & 1-3\gamma-\lambda \end{vmatrix} = 0$

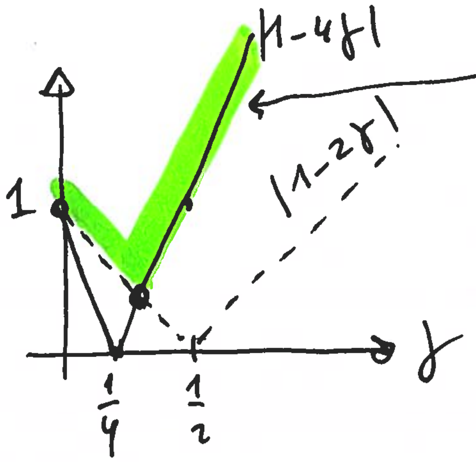
max. 5pt.

$\iff \lambda = 1 - 4\gamma$ of $\lambda = 1 - 2\gamma$

$|1-4\gamma| < 1$ èn $|1-2\gamma| < 1$

$\iff \gamma < \frac{1}{2}$ èn $\gamma < 1$ $\Rightarrow$ $\gamma < \frac{1}{2}$ convergentie

c)



$\rho(I - \gamma A) = \max\{|1-4\gamma|, |1-2\gamma|\}$

max. 5pt.

snelste convergentie voor laagste waarde van : $-(1-4\gamma) = 1 - 2\gamma$

$\iff \gamma = 1/3$

max. 15 pt.

<u>vraag 3</u>

$$\varphi(x) = x - \frac{x^4 - \alpha}{2\alpha^3} \quad , \alpha > 1$$

$x_0 \in (0, \alpha)$

a) $$\varphi'(x) = 1 - \frac{4x^3}{2\alpha^3} = 1 - \frac{2x^3}{\alpha^3}$$

$$\left(\varphi'(\alpha^{1/4}) = 1 - 2\frac{\alpha^{3/4}}{\alpha^3} \neq 0 \right)$$

$$|\varphi'(x)| = \left|1 - \frac{2x^3}{\alpha^3}\right| < 1$$

<u>als</u> $-1 < 1 - \frac{2x^3}{\alpha^3} < 1$

max. 8 pt.

$$\Leftrightarrow -2 < \underbrace{-\frac{2x^3}{\alpha^3}}_{<0} < 0 \quad \checkmark \text{ ok}$$

$$\Leftrightarrow \frac{2x^3}{\alpha^3} < 2$$

$$\Leftrightarrow x < \alpha \qquad (x_0 < \alpha \Rightarrow \text{ok})$$

b) $\varphi'(\alpha^{1/4}) \neq 0 \Rightarrow$ lineaire convergentie

$$\varphi'(\alpha^{1/4}) = 0 \text{ als } 1 - \frac{2\alpha^{3/4}}{\alpha^3} = 0$$

max. 7 pt.

$$\Leftrightarrow 2\alpha^{3/4} = \alpha^3$$

$$\Leftrightarrow \alpha = \sqrt[9]{16}$$

(dan kwadratische convergentie)

$$\varphi''(x) = -\frac{6x^2}{\alpha^3} \neq 0$$ (dus nooit kubische convergentie, want $x_0 > 0$)

max. 20 pt.

vraag 4

$$f''(x) \approx \frac{2f(x-h) - 3f(x) + f(x+2h)}{3h^2}, \quad h>0$$

a) $f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) \; --$

$f(x+2h) = f(x) + 2h f'(x) + \frac{4h^2}{2} f''(x) + \frac{8h^3}{6} f'''(x \; --$

$$\Rightarrow \frac{2\,f(x-h) - 3f(x) + f(x+2h)}{3h^2}$$

$$= \left\{ \begin{array}{l} \cancel{2f(x)} - \cancel{2h f'(x)} + \frac{2h^2}{2} f''(x) - \frac{2h^3}{6} f'''(x) \; -- \\ - \cancel{3f(x)} + \cancel{f(x)} + \cancel{2h f'(x)} + \frac{4h^2}{2} f''(x) + \frac{8}{6} h^3 f'''(x) \; --- \end{array} \right\} \Big/ (3h^2)$$

$$= f''(x) - \frac{1}{9} h f'''(x) + \frac{4}{9} h f'''(x) \; --$$

$$= f''(x) + \frac{1}{3} h^1 f'''(x)$$

$$\Rightarrow c = -\frac{1}{3} \quad (\text{"}+\tfrac{1}{3}\text{"} --) \quad \text{en } q = 1$$

max. 10 pt

b) $f(x) = \sin(5x)$

$f'(x) = 5\cos(5x)$

$f''(x) = -25 \sin(5x)$

$f'''(x) = -125 \cos(5x)$

$$|\text{fout}| = \frac{1}{3} \cdot h \cdot 125 \cdot \underbrace{|\cos(5x)|}_{\leq 1} < \varepsilon$$

$$\Rightarrow h < \frac{3\varepsilon}{125}$$

max. 10 pt.

vraag 5

max. 20 pt.

$$I = \int_0^1 f(x)dx \approx T_1 = f\left(\tfrac{1}{2}\right)$$

$$R_1 = I - T_1 = \frac{f''(\eta_1)}{24}$$

a) $$I = \int_0^{1/2} f + \int_{1/2}^1 f\,dx = \underbrace{\tfrac{1}{2}f\left(\tfrac{1}{4}\right) + \tfrac{1}{2}f\left(\tfrac{3}{4}\right)}_{= T_2} + \left(\tfrac{1}{2}\right)^3 \frac{f''(\xi)}{24} + \left(\tfrac{1}{2}\right)^3 \frac{f''(\xi')}{24}$$

$$\Rightarrow I - T_2 = \frac{1}{96} f''(\xi_2)$$

(tussenwaarde stelling $f''(\xi_2) = \dfrac{f''(\xi) + f''(\xi')}{2}$

max. 10 pt.

b) $$I_3 = c_1 T_1 + c_2 T_2$$

$p(x) = ax^2 + bx + c$, $p'(x) = 2ax + b$, $p''(x) = 2a \overset{\text{noem}}{=} C$ constante

$I - T_1 = \dfrac{C}{24}$, $T_1 = I - \dfrac{C}{24}$

$I - T_2 = \dfrac{C}{96}$, $T_2 = I - \dfrac{C}{96}$

$$\Rightarrow I - T_3 = I - c_1 T_1 - c_2 T_2 = I - c_1\left(I - \tfrac{C}{24}\right) - c_2\left(I - \tfrac{C}{96}\right)$$

$$= (1 - (c_1 + c_2)) I + \frac{C}{96}(4c_1 + c_2)$$

exact voor $p(x)$: $\begin{cases} 1 - (c_1 + c_2) = 1 \\ 4c_1 + c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 1 \\ 4c_1 + c_2 = 0 \end{cases}$

$$\Rightarrow c_1 = -\tfrac{1}{3},\ c_2 = \tfrac{4}{3}$$

en dus $T_3 = -\tfrac{1}{3} T_1 + \tfrac{4}{3} T_2$

max. 10 pt.

max. 20 pt.

vraag 6

$$y_{n+1} = y_n + \Delta t\,(\theta\, f(y_n) + (1-\theta) f(y_{n+1}))$$

a) $\theta = 0$: Euler-Backward

$\theta = 1$: Euler-Forward

max. 5pt

b) $y' = \lambda y$

$\Rightarrow y_{n+1} = y_n + \Delta t\,\theta \lambda y_n + \Delta t\,(1-\theta)\lambda y_{n+1}$

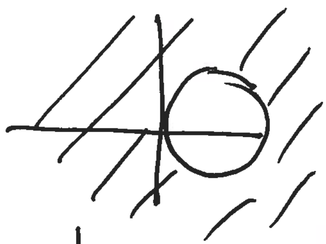
$\Rightarrow (1-(1-\theta)\Delta t\lambda) y_{n+1} = (1+\theta\Delta t\lambda) y_n$

max. 5 pt.

stab. criterium: $\left|\dfrac{1+\theta\Delta t\lambda}{1-(1-\theta)\Delta t\lambda}\right| < 1$

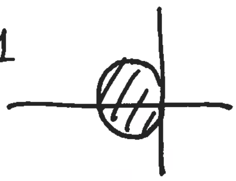
of (met $z \overset{def}{=} \lambda\Delta t$): $\left|\dfrac{1+\theta z}{1-(1-\theta)z}\right| < 1$

c) $\theta = 0$: $\left|\dfrac{1}{1-z}\right| < 1$



max. 5 pt.

$\theta = 1$: $|1+z| < 1$



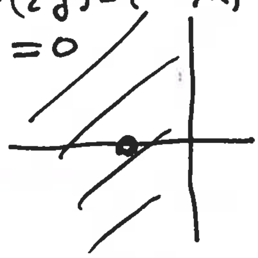
$\theta = \frac{1}{2}$: $|1+\frac{1}{2}z| < |1-\frac{1}{2}z|$

"=" : $|1+\frac{1}{2}(x+iy)| = |1-\frac{1}{2}(x+iy)| \Leftrightarrow (1+\frac{1}{2}x)^2 + (\frac{1}{2}y)^2 = (1-\frac{1}{2}x)^2 + (\frac{1}{2}y)^2$

en $z = x+iy$ $\Leftrightarrow x = 0$

voor $x=-1, y=0$: $z=-1$

$|1-\frac{1}{2}| = \frac{1}{2} < |1+\frac{1}{2}| = \frac{3}{2}$



d)

vervolg Vraag 6 d)

Taylor: $y(b) = y(a) + (b-a)y'(a) + \frac{(b-a)^2}{2} y''(a) + \frac{(b-a)^3}{6} y'''(a) + O((b-a)^3)$

max. 5 pt.

combineer Taylor met $\begin{cases} a = t \\ b = t+\Delta t \end{cases}$

en $\begin{cases} a = t+\Delta t \\ b = t \end{cases}$

$\Rightarrow y(t+\Delta t) = y(t) + \Delta t \left(\theta y'(t) + (1-\theta) y'(t+\Delta t) + \cdots\right)$

gebruik $y' = f(y)$

$\Rightarrow$ lokale afbreekfout τ_n:

$$\tau_n = \frac{(\Delta t)^2}{2}\left\{\theta y''(t_n) - (1-\theta) y''(t_{n+1})\right\} + \frac{(\Delta t)^3}{6}\left\{\theta y'''(t_n) + (1-\theta) y'''(t_{n+1})\right\} + O((\Delta t)^4)$$

Taylor $y''(t_{n+1})$ en $y'''(t_{n+1})$ rond t_n

levert op: $\tau_n = \underbrace{\frac{(2\theta-1)(\Delta t)^2}{2} y''(t_n)}_{=0} + \underbrace{\frac{(3\theta-2)(\Delta t)^3}{6} y'''(t_n)}_{\neq 0} + O((\Delta t)^4)$

voor $\theta \neq \frac{1}{2}$: $O((\Delta t)^2)$

voor $\theta = \frac{1}{2}$: $O((\Delta t)^3)$