Utrecht University
Mathematical Institute

Answers, Exam for Introduction to Financial Mathematics, WISB373

Wednesday June 30th 2021, 15:15-18:15 o'clock **(3 hours examination)**

1. Apply the Itô-Doeblin formula to $2^{W(t)}$, where $\{W(t) : t \geq 0\}$ is a standard Brownian motion. Is this a martingale? (10 pts.)

Answer 1: With $g(t) = 2^{W(t)}$, we find,

$$\mathrm{d}g(t) = \ln 2 \cdot 2^{W(t)}\mathrm{d}W(t) + \frac{(\ln 2)^2}{2}2^{W(t)}\mathrm{d}t.$$

Note that g, g_x, g_{xx} exist and are continuous!

Due to the appearance of a dt-term, the process is not a martingale.

2. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$, with

$$S(t) = S(0) \exp\left\{\int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u\right\}.$$

(a) Let

$$X(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u$$

and determine the distribution of $X(t)$. (10 pts.)

(b) Prove that $\{S(t) : t \geq 0\}$ is an Itô process. (10 pts.)

(c) Let r be a constant interest rate. Find the risk-neutral measure $\tilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ (i.e. $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0,\ A \in \mathcal{F}$), such that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$. (10 pts.)

Proof 2(a) : Let $Y(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u)$. Since $Y(t)$ is the Itô integral of a deterministic process, by Theorem 4.4.9, $Y(t)$ is normally distributed with $\mathbb{E}[Y(t)] = 0$ and

$$\mathbb{V}\mathrm{ar}[Y(t)] = \int_0^t \mathrm{e}^{-2u}\mathrm{d}u = \frac{1}{2}(1 - \mathrm{e}^{-2t}).$$

Since

$$X(t) = Y(t) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u = Y(t) + t + \frac{1}{4}(\mathrm{e}^{-2t} - 1),$$

we see that $X(t)$ is normally distributed, with mean

$$\mathbb{E}[X(t)] = t + \frac{1}{4}(\mathrm{e}^{-2t} - 1)$$

and variance

$$\mathbb{V}\mathrm{ar}[X(t)] = \mathbb{V}\mathrm{ar}[Y(t)] = \frac{1}{2}(1 - \mathrm{e}^{-2t}).$$

Proof 2(b) : With

$$X(t)=\int_0^t \mathrm{e}^{-u}\mathrm{d}W(u)+\int_0^t (1-\frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t,$$

we have

$$\mathrm{d}X(t)=\mathrm{e}^{-t}\mathrm{d}W(t)+(1-\frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t,$$

and $\mathrm{d}X(t)\mathrm{d}X(t)=\mathrm{e}^{-2t}\mathrm{d}t$. Note that $S(t)=S(0)\mathrm{e}^{X(t)}$, so let $f(x)=S(0)\mathrm{e}^{x}$, then $f_x(x)=f_{xx}(x)=f(x)$. By the Itô-Doeblin formula, we have,

$$\begin{aligned}
\mathrm{d}S(t) &= \mathrm{d}f(X(t))=S(t)\mathrm{d}X(t)+\frac{1}{2}S(t)\mathrm{d}X(t)\mathrm{d}X(t)\\
&= S(t)\left(\mathrm{e}^{-t}\mathrm{d}W(t)+(1-\frac{1}{2}\mathrm{e}^{-2t})\mathrm{d}t\right)+\frac{1}{2}S(t)\mathrm{e}^{-2t}\mathrm{d}t\\
&= S(t)\mathrm{d}t+S(t)\mathrm{e}^{-t}\mathrm{d}W(t).
\end{aligned}$$

This shows that

$$S(t)=S(0)+\int_0^t S(u)\mathrm{d}u+\int_0^t S(u)\mathrm{e}^{-u}\mathrm{d}W(u),$$

hence $\{S(t):t\geq 0\}$ in an Itô process.

Proof 2(c) : Define

$$\theta(t)=\frac{1-r}{\mathrm{e}^{-t}}=\mathrm{e}^{t}(1-r).$$

Consider the random variable Z, defined by

$$\begin{aligned}
Z &= \exp\left(-\int_0^T \theta(u)\mathrm{d}W(u)-\frac{1}{2}\int_0^T \theta^2(u)\mathrm{d}u\right)\\
&= \exp\left(-\int_0^T \mathrm{e}^{t}(1-r)\mathrm{d}W(u)-\frac{1}{2}\int_0^T \mathrm{e}^{2u}(1-r)\mathrm{d}u\right).
\end{aligned}$$

Note that $\int_0^t \theta(u)\mathrm{d}W(u)$, $\int_0^t \theta^2(u)\mathrm{d}u$ and θ are continuous functions on the compact interval $[0,T]$, hence they are all bounded. This implies that $\mathbb{E}\left[\int_0^T \theta^2(u)Z^2(u)\mathrm{d}u\right]<\infty$. Define the measure $\tilde{\mathbb{P}}$ on $\mathcal{F}$ by $\tilde{\mathbb{P}}(A)=\int_A Z\mathrm{d}\mathbb{P}$ and consider the process $\{\tilde{W}(t):0\leq t\leq T\}$ with

$$\tilde{W}(t)=\int_0^t \theta(u)\mathrm{d}u+W(t)=\int_0^t \mathrm{e}^{u}(1-r)\mathrm{d}u+W(t)=(1-r)(\mathrm{e}^{t}-1)+W(t).$$

By Girsanov's Theorem, the process $\{\tilde{W}(t):0\leq t\leq T\}$ is a Brownian motion under $\tilde{\mathbb{P}}$ and hence it is a martingale under $\tilde{\mathbb{P}}$. Using the SDE obtained in part (a), together with the Itô product rule, we have

$$\begin{aligned}
\mathrm{d}(\mathrm{e}^{-rt}S(t)) &= \mathrm{e}^{-rt}\mathrm{d}S(t)-r\mathrm{e}^{-rt}S(t)\mathrm{d}t\\
&= \mathrm{e}^{-rt}\left(S(t)\mathrm{d}t+S(t)\mathrm{e}^{-t}\mathrm{d}W(t)\right)-r\mathrm{e}^{-rt}S(t)\mathrm{d}t\\
&= \mathrm{e}^{-rt}S(t)\left((1-r)\mathrm{d}t+\mathrm{e}^{-t}\mathrm{d}W(t)\right)\\
&= \mathrm{e}^{-rt}S(t)\left(\mathrm{e}^{-t}\theta(t)\mathrm{d}t+\mathrm{e}^{-t}\mathrm{d}W(t)\right)\\
&= \mathrm{e}^{-t(r+1)}S(t)\mathrm{d}\tilde{W}(t).
\end{aligned}$$

Since $\mathrm{e}^{-rt}S(t)$ is an Itô integral, we see that the discounted price process is a martingale under $\tilde{\mathbb{P}}$.

3. Suppose that $X(t)$ satisfies the following Stochastic Differential Equation (SDE):

$$\mathrm{d}X(t) = 0.04X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),$$

and $Y(t)$ satisfies:

$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.1Y(t)\mathrm{d}W(t).$$

Parameters β, σ are positive constants and both processes are driven by the same Brownian Motion $W(t)$.
For a given process

$$Z(t) = 2\frac{X(t)}{Y(t)} - \lambda t,$$

with $\lambda \in \mathbb{R}^+$.

a. Find the SDE for $Z(t)$. (10 pts.)

b. For which values of β and λ is process $Z(t)$ a martingale? (10 pts.)

Answer 3*a*. We have:

$$X(t) = e^{\sigma W(t) - \frac{\sigma^2}{2}t + 0.04t},$$
$$\mathrm{d}X(t) = 0.04X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t).$$

$$Y(t) = e^{0.1W(t) - \frac{0.01}{2}t + \beta t},$$
$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.1Y(t)\mathrm{d}W(t).$$

Using the expressions for $X(t)$ and $Y(t)$, we get,

$$Z(t) = 2e^{(\sigma - 0.1)W(t) + (0.04 + \frac{0.01}{2} - \beta - \frac{\sigma^2}{2})t} - \lambda t.$$

Answer 3*b*. A martingale process does not contain a drift term. We have,

$$\mathrm{d}Z(t) = (Z + \lambda t)(0.01 + 0.04 - \beta - 0.1\sigma)\mathrm{d}t - \lambda \mathrm{d}t + (Z + \lambda t)(\sigma - 0.1)\mathrm{d}W(t).$$

With β and σ constant, and $\lambda \in \mathbb{R}^+$, the necessary conditions for a vanishing drift term are $\lambda = 0$ and

$$0.01 + 0.04 - \beta - 0.1\sigma = 0 \implies \beta = 0.05 - 0.15\sigma.$$

To check this result we employ the Itô's derivative rules for multivariate functions, i.e.,

$$\begin{aligned}
\mathrm{d}Z(t) &= 2\left(\frac{\mathrm{d}X(t)}{Y(t)} - \frac{X(t)\mathrm{d}Y(t)}{Y^2(t)} - \frac{\mathrm{d}X(t)\mathrm{d}Y(t)}{Y^2} + \frac{X(t)\mathrm{d}Y^2(t)}{Y^3}\right) - \lambda \mathrm{d}t \\
&= (Z(t) + \lambda t)\big((0.04 - \beta - 0.1\sigma + 0.01)\mathrm{d}t + (\sigma - 0.1)\mathrm{d}W(t)\big) - \lambda \mathrm{d}t,
\end{aligned}$$

which yields the same constraints. Hence, $\lambda = 0$ and $\beta = 0.05 - 0.1\sigma$.

4. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDEs given by

$$\begin{aligned}
\mathrm{d}S_1(t) &= 2S_1(t)\mathrm{d}W_1(t) + 3S_1(t)\mathrm{d}W_2(t), \\
\mathrm{d}S_2(t) &= S_2(t)\mathrm{d}t + S_2(t)\mathrm{d}W_1(t).
\end{aligned}$$

(a) Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

(b) Consider a finite time T (expiration date), and suppose the interest rate is a constant, i.e. $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t)) : 0 \leq t \leq T\}$. (10 pts.)

Proof 4(a) : We apply the Itô product rule, we find

$$\mathrm{d}(S_1(t)S_2(t)) = S_1(t)\mathrm{d}S_2(t) + S_2(t)\mathrm{d}S_1(t) + \mathrm{d}S_1(t)\mathrm{d}S_2(t).$$

Using

$$\mathrm{d}S_1(t) = 2S_1(t)\mathrm{d}W_1(t) + 3S_1(t)\mathrm{d}W_2(t), \mathrm{d}S_2(t) = S_2(t)\mathrm{d}t + S_2(t)\mathrm{d}W_1(t)$$

and

$$\mathrm{d}S_1(t)\mathrm{d}S_2(t) = 2S_1(t)S_2(t)\mathrm{d}t,$$

we get after simplifying,

$$\mathrm{d}(S_1(t)S_2(t)) = 3S_1(t)S_2(t)\mathrm{d}t + 3S_1(t)S_2(t)\mathrm{d}W_1(t) + 3S_1(t)S_2(t)\mathrm{d}W_2(t).$$

Equivalently,

$$\begin{aligned} S_1(t)S_2(t) &= S_1(0)S_2(0) + \int_0^t 3S_1(u)S_2(u)\mathrm{d}u + \int_0^t 3S_1(u)S_2(u)\mathrm{d}W_1(u) \\ &+ \int_0^t 3S_1(u)S_2(u)\mathrm{d}W_2(u) \end{aligned}$$

Hence, $\{S_1(t)S_2(t) : t \geq 0\}$ is a two-dimensional Itô process.

Proof 4(b) : Using the notation of the book, we have $\alpha_1 = 0, \sigma_{11} = 2, \sigma_{12} = 3, \alpha_2 = 1, \sigma_{21} = 1, \sigma_{22} = 0$. The market price equations in this case are given by the system,

$$\begin{aligned} -r &= 2\theta_1(t) + 3\theta_2(t) \\ 1 - r &= \theta_1(t). \end{aligned}$$

Solving for $\theta_1(t), \theta_2(t)$, we get

$$\begin{aligned} \theta_1(t) &= 1 - r \\ \theta_2(t) &= \frac{r-2}{3}. \end{aligned}$$

Setting,

$$\begin{aligned} Z &= \exp\left\{-\int_0^T (\theta_1(t)\mathrm{d}W_1(t) + \theta_2(t)\mathrm{d}W_2(t)) - \frac{1}{2}\int_0^T \left(\theta_1^2(t) + \theta_2^2(t)\right)\mathrm{d}t\right\} \\ &= \exp\left\{(r-1)W_1(T) + \frac{2-r}{3}W_2(T) - \frac{1}{2}\left((1-r)^2 + \frac{(r-2)^2}{9}\right)T\right\}, \end{aligned}$$

the risk-neutral measure is given by $\tilde{\mathbb{P}}(A) = \int_A Z\mathrm{d}\mathbb{P}$. To check this, we set $\tilde{W}_1(t) = (1-r)t + W_1(t)$ and $\tilde{W}_2(t) = \frac{r-2}{3}t + W_2(t)$. By the 2-dimensional Girsanov Theorem, the process $\{(\tilde{W}_1(t), \tilde{W}_2(t) : 0 \leq t \leq T\}$ is a 2-dimensional Brownian motion under $\tilde{\mathbb{P}}$. Rewriting

$e^{-rt}S_1(t), e^{-rt}S_2(t)$ in terms of $\tilde{W}_1(t), \tilde{W}_2(t)$, we get, after applying the Itô product rule,

$$\begin{aligned} d(e^{-rt}S_1(t)) &= e^{-rt}S_1(t)(2d\tilde{W}_1(t) + 3d\tilde{W}_2(t)) \\ d(e^{-rt}S_2(t)) &= e^{-rt}S_2(t)d\tilde{W}_1(t), \end{aligned}$$

which shows that the discounted price processes are Itô integrals, and hence martingales under $\tilde{\mathbb{P}}$.

5. Assume we have a European call and a put option, with the same expiry date $T = 1/4$, i.e., exercise in three months, and strike price $K = 10$ Euro. The current share price is 11 Euro, assuming a constant interest rate $r = 6\%$. Determine an arbitrage opportunity if both options currently have the value $c(0) = p(0) = 2.5$ Euro. (10 pts.)

Answer 5. We form two portfolios using the options, the underlying asset and a cash amount K, with one based on the put $p(t)$ and the other based on the call $c(t)$, as follows,

$$\Pi_1(t) = p(t) + S(t),$$
$$\Pi_2(t) = c(t) + Ke^{-0.06(0.25-t)}.$$

These portfolios have same value at expiry time T. By the put-call parity, their value should be equal any time prior to the exercise time, as otherwise arbitrage opportunities will appear. In the case of a mismatch in value, one can buy the cheaper portfolio and sell the expensive one. At the expiry time T, one can trade these two portfolios without any cost, hence the initial sell-buy difference is reflected as a profit. Returning to the exercise and looking at the arbitrage opportunity when both options are worth 2.5 €, we assume this takes place at $t < T$. Using the put-call parity relation, we find the following relation for not having an arbitrage opportunity,

$$S(t) = 10e^{-0.06(0.25-t)}.$$

Hence, at $t = 0$, assuming that the option values are 2.5 €, one can benefit from selling portfolio Π_1 and buying Π_2. As long as $S(t) > 10e^{-0.06(0.25-t)}$, one can follow this strategy, when $S(t) < 10e^{-0.06(0.25-t)}$, one should revert the strategy.