Answers, Exam for Introduction to Financial Mathematics, WISB373

Wednesday June 30th 2021, 15:15-18:15 o’clock (3 hours examination)

1. Apply the Itô-Doeblin formula to $2^W(t)$, where $\{W(t) : t \geq 0\}$ is a standard Brownian motion. Is this a martingale? (10 pts.)

**Answer 1:** With $g(t) = 2^W(t)$, we find,
\[
dg(t) = \ln 2 \cdot 2^W(t) \, dW(t) + \frac{(\ln 2)^2}{2} \cdot 2^W(t) \, dt.
\]

Note that $g, g_x, g_{xx}$ exist and are continuous!

Due to the appearance of a $dt$-term, the process is not a martingale.

2. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$, with $S(t) = S(0) \exp\left\{ \int_0^t e^{-u}dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u}) \, du \right\}$.

(a) Let
\[
X(t) = \int_0^t e^{-u}dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u}) \, du
\]
and determine the distribution of $X(t)$. (10 pts.)

(b) Prove that $\{S(t) : t \geq 0\}$ is an Itô process. (10 pts.)

(c) Let $r$ be a constant interest rate. Find the risk-neutral measure $\tilde{P}$, equivalent to $P$ (i.e. $\tilde{P}(A) = 0$ if and only if $P(A) = 0$, $A \in \mathcal{F}$), such that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{P}$. (10 pts.)

**Proof 2(a):** Let $Y(t) = \int_0^t e^{-u}dW(u)$. Since $Y(t)$ is the Itô integral of a deterministic process, by Theorem 4.4.9, $Y(t)$ is normally distributed with $\mathbb{E}[Y(t)] = 0$ and
\[
\text{Var}[Y(t)] = \int_0^t e^{-2u} \, du = \frac{1}{2}(1 - e^{-2t}).
\]

Since
\[
X(t) = Y(t) + \int_0^t (1 - \frac{1}{2}e^{-2u}) \, du = Y(t) + t + \frac{1}{4}(e^{-2t} - 1),
\]
we see that $X(t)$ is normally distributed, with mean
\[
\mathbb{E}[X(t)] = t + \frac{1}{4}(e^{-2t} - 1)
\]
and variance
\[
\text{Var}[X(t)] = \text{Var}[Y(t)] = \frac{1}{2}(1 - e^{-2t}).
\]
Proof 2(b) : With

\[ X(t) = \int_0^t e^{-u}dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2t})dt, \]

we have

\[ dX(t) = e^{-t}dW(t) + (1 - \frac{1}{2}e^{-2t})dt, \]

and \( dX(t)dX(t) = e^{-2t}dt \). Note that \( S(t) = S(0)e^{X(t)} \), so let \( f(x) = S(0)e^x \), then \( f_x(x) = f_{xx}(x) = f(x) \). By the Itô-Doeblin formula, we have,

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= df(X(t)) = S(t)dX(t) + \frac{1}{2}S(t)dX(t)dX(t) \\
&= S(t)\left(e^{-t}dW(t) + (1 - \frac{1}{2}e^{-2t})dt\right) + \frac{1}{2}S(t)e^{-2t}dt \\
&= S(t)dt + S(t)e^{-t}dW(t).
\end{align*}
\]

This shows that

\[ S(t) = S(0) + \int_0^t S(u)du + \int_0^t S(u)e^{-u}dW(u), \]

hence \( \{S(t) : t \geq 0\} \) in an Itô process.

Proof 2(c) : Define

\[ \theta(t) = \frac{1 - r}{e^{-t}} = e^t(1 - r). \]

Consider the random variable \( Z \), defined by

\[
Z = \exp\left(-\int_0^T \theta(u)dW(u) - \frac{1}{2} \int_0^T \theta^2(u)du\right).
\]

Note that \( \int_0^t \theta(u)dW(u) \) and \( \theta \) are continuous functions on the compact interval \([0, T] \), hence they are all bounded. This implies that \( \mathbb{E}\left[\int_0^T \theta^2(u)du\right] < \infty \). Define the measure \( \bar{P} \) on \( \mathcal{F} \) by \( \bar{P}(A) = \int_A Zd\mathbb{P} \) and consider the process \( \{\bar{W}(t) : 0 \leq t \leq T\} \) with

\[
\bar{W}(t) = \int_0^t \theta(u)du + W(t) = \int_0^t e^u(1-r)du + W(t) = (1-r)(e^t-1)+W(t).
\]

By Girsanov’s Theorem, the process \( \{\bar{W}(t) : 0 \leq t \leq T\} \) is a Brownian motion under \( \bar{P} \) and hence it is a martingale under \( \bar{P} \). Using the SDE obtained in part (a), together with the Itô product rule, we have

\[
\begin{align*}
d(e^{-rt}S(t)) &= e^{-rt}dS(t) - re^{-rt}S(t)dt \\
&= e^{-rt}\left(S(t)dt + S(t)e^{-t}dW(t)\right) - re^{-rt}S(t)dt \\
&= e^{-rt}S(t)\left((1-r)dt + e^{-t}dW(t)\right) \\
&= e^{-rt}S(t)\left(e^{-t}\theta(t)dt + e^{-t}dW(t)\right) \\
&= e^{-t(r+1)}S(t)d\bar{W}(t).
\end{align*}
\]

Since \( e^{-rt}S(t) \) is an Itô integral, we see that the discounted price process is a martingale under \( \bar{P} \).
3. Suppose that $X(t)$ satisfies the following Stochastic Differential Equation (SDE):
\[ dX(t) = 0.04X(t)dt + \sigma X(t)dW(t), \]
and $Y(t)$ satisfies:
\[ dY(t) = \beta Y(t)dt + 0.1Y(t)dW(t). \]
Parameters $\beta$, $\sigma$ are positive constants and both processes are driven by the same Brownian Motion $W(t)$.
For a given process
\[ Z(t) = 2\frac{X(t)}{Y(t)} - \lambda t, \]
with $\lambda \in \mathbb{R}^+$.

a. Find the SDE for $Z(t)$. (10 pts.)
b. For which values of $\beta$ and $\lambda$ is process $Z(t)$ a martingale? (10 pts.)

**Answer 3a.** We have:
\[ X(t) = e^{\sigma W(t) - \frac{\sigma^2}{2} t + 0.04t}, \]
\[ dX(t) = 0.04X(t)dt + \sigma X(t)dW(t). \]
\[ Y(t) = e^{0.1W(t) - \frac{0.01}{2} t + \beta t}, \]
\[ dY(t) = \beta Y(t)dt + 0.1Y(t)dW(t). \]
Using the expressions for $X(t)$ and $Y(t)$, we get,
\[ Z(t) = 2e^{(\sigma - 0.1)W(t) + (0.04 + \frac{0.01}{2} - \beta - \frac{\sigma^2}{2})t - \lambda t}. \]

**Answer 3b.** A martingale process does not contain a drift term. We have,
\[ dZ(t) = (Z + \lambda t)(0.01 + 0.04 - \beta - 0.1\sigma)dt - \lambda dt + (Z + \lambda t)(\sigma - 0.1)dW(t). \]
With $\beta$ and $\sigma$ constant, and $\lambda \in \mathbb{R}^+$, the necessary conditions for a vanishing drift term are $\lambda = 0$ and
\[ 0.01 + 0.04 - \beta - 0.1\sigma = 0 \implies \beta = 0.05 - 0.15\sigma. \]
To check this result we employ the Itô’s derivative rules for multivariate functions, i.e.,
\[ dZ(t) = 2 \left( \frac{dX(t)}{Y(t)} - \frac{X(t)dY(t)}{Y^2(t)} - \frac{dX(t)dY(t)}{Y^2} + \frac{X(t)dY^2(t)}{Y^3} \right) - \lambda dt \]
\[ = (Z(t) + \lambda t)((0.04 - \beta - 0.1\sigma + 0.01)dt + (\sigma - 0.1)dW(t)) - \lambda dt, \]
which yields the same constraints. Hence, $\lambda = 0$ and $\beta = 0.05 - 0.1\sigma$.

4. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDEs given by
\[ dS_1(t) = 2S_1(t)dW_1(t) + 3S_1(t)dW_2(t), \]
\[ dS_2(t) = S_2(t)dt + S_2(t)dW_1(t). \]
(a) Show that \{S_1(t)S_2(t) : t \geq 0\} is a 2-dimensional Itô-process. (10 pts.)

(b) Consider a finite time \(T\) (expiration date), and suppose the interest rate is a constant, i.e. \(R(t) = r\) for all \(t > 0\). Show that the market price equations have a unique solution, and determine the risk-neutral probability measure \(\tilde{\mathbb{P}}\) for the process \\{(S_1(t), S_2(t)) : 0 \leq t \leq T\}. (10 pts.)

**Proof 4(a)**: We apply the Itô product rule, we find

\[
d(S_1(t)S_2(t)) = S_1(t)dS_2(t) + S_2(t)dS_1(t) + dS_1(t)dS_2(t).
\]

Using

\[
dS_1(t) = 2S_1(t)dW_1(t) + 3S_1(t)dW_2(t),
\]

\[
dS_2(t) = S_2(t)dt + S_2(t)dW_1(t)
\]

and

\[
dS_1(t)dS_2(t) = 2S_1(t)S_2(t)dt,
\]

we get after simplifying,

\[
d(S_1(t)S_2(t)) = 3S_1(t)S_2(t)dt + 3S_1(t)S_2(t)dW_1(t) + 3S_1(t)S_2(t)dW_2(t).
\]

Equivalently,

\[
S_1(t)S_2(t) = S_1(0)S_2(0) + \int_0^t 3S_1(u)S_2(u)du + \int_0^t 3S_1(u)S_2(u)dW_1(u) + \int_0^t 3S_1(u)S_2(u)dW_2(u)
\]

Hence, \{S_1(t)S_2(t) : t \geq 0\} is a two-dimensional Itô process.

**Proof 4(b)**: Using the notation of the book, we have \(\alpha_1 = 0, \sigma_{11} = 2, \sigma_{12} = 3, \alpha_2 = 1, \sigma_{21} = 1, \sigma_{22} = 0\). The market price equations in this case are given by the system,

\[
\begin{align*}
-r &= 2\theta_1(t) + 3\theta_2(t) \\
1 - r &= \theta_1(t).
\end{align*}
\]

Solving for \(\theta_1(t), \theta_2(t)\), we get

\[
\begin{align*}
\theta_1(t) &= 1 - r \\
\theta_2(t) &= \frac{r - 2}{3}.
\end{align*}
\]

Setting,

\[
Z = \exp\left\{ - \int_0^T (\theta_1(t)dW_1(t) + \theta_2(t)dW_2(t)) - \frac{1}{2} \int_0^T (\theta_1^2(t) + \theta_2^2(t)) dt \right\}
\]

\[
= \exp\left\{ (r - 1)W_1(T) + \frac{2 - r}{3}W_2(T) - \frac{1}{2} \left( (r - 1)^2 + \frac{(r - 2)^2}{9} \right) T \right\},
\]

the risk-neutral measure is given by \(\tilde{\mathbb{P}}(A) = \int_A Zd\mathbb{P}\). To check this, we set \(W_1(t) = (1 - r)t + W_1(t)\) and \(W_2(t) = \frac{r - 2}{3}t + W_2(t)\). By the 2-dimensional Girsanov Theorem, the process \{(\tilde{W}_1(t), \tilde{W}_2(t)) : 0 \leq t \leq T\} is a 2-dimensional Brownian motion under \(\tilde{\mathbb{P}}\). Rewriting
e^{-rt}S_1(t), e^{-rt}S_2(t) in terms of \( \tilde{W}_1(t), \tilde{W}_2(t) \), we get, after applying the Itô product rule,

\[
\begin{align*}
\text{d}(e^{-rt}S_1(t)) &= e^{-rt}S_1(t)(2d\tilde{W}_1(t) + 3d\tilde{W}_2(t)) \\
\text{d}(e^{-rt}S_2(t)) &= e^{-rt}S_2(t)d\tilde{W}_1(t),
\end{align*}
\]

which shows that the discounted price processes are Itô integrals, and hence martingales under \( \tilde{P} \).

5. Assume we have a European call and a put option, with the same expiry date \( T = 1/4 \), i.e., exercise in three months, and strike price \( K = 10 \) Euro. The current share price is 11 Euro, assuming a constant interest rate \( r = 6\% \). Determine an arbitrage opportunity if both options currently have the value \( c(0) = p(0) = 2.5 \) Euro. (10 pts.)

Answer 5. We form two portfolios using the options, the underlying asset and a cash amount \( K \), with one based on the put \( p(t) \) and the other based on the call \( c(t) \), as follows,

\[
\begin{align*}
\Pi_1(t) &= p(t) + S(t), \\
\Pi_2(t) &= c(t) + Ke^{-0.06(0.25-t)}.
\end{align*}
\]

These portfolios have same value at expiry time \( T \). By the put-call parity, their value should be equal any time prior to the exercise time, as otherwise arbitrage opportunities will appear. In the case of a mismatch in value, one can buy the cheaper portfolio and sell the expensive one. At the expiry time \( T \), one can trade these two portfolios without any cost, hence the initial sell-buy difference is reflected as a profit. Returning to the exercise and looking at the arbitrage opportunity when both options are worth 2.5 Euro, we assume this takes place at \( t < T \). Using the put-call parity relation, we find the following relation for not having an arbitrage opportunity,

\[
S(t) = 10e^{-0.06(0.25-t)}.
\]

Hence, at \( t = 0 \), assuming that the option values are 2.5 Euro, one can benefit from selling portfolio \( \Pi_1 \) and buying \( \Pi_2 \). As long as \( S(t) > 10e^{-0.06(0.25-t)} \), one can follow this strategy, when \( S(t) < 10e^{-0.06(0.25-t)} \), one should revert the strategy.