Utrecht University
Mathematical Institute

Final Exam for Introduction to Financial Mathematics, WISB373

Wednesday June 30th 2021, 15:15-18:15 o'clock **(3 hours examination)**

1. Apply the Itô-Doeblin formula to $2^{W(t)}$, where $\{W(t) : t \geq 0\}$ is a Brownian motion. Is this a martingale? (10 pts.)

2. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$, with

$$S(t) = S(0) \exp\left\{\int_0^t \mathrm{e}^{-u} \mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u}) \mathrm{d}u\right\}.$$

(a) Let

$$X(t) = \int_0^t \mathrm{e}^{-u} \mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u}) \mathrm{d}u$$

and determine the distribution of $X(t)$. (10 pts.)

(b) Prove that $\{S(t) : t \geq 0\}$ is an Itô process. (10 pts.)

(c) Let r be a constant interest rate. Find the risk-neutral measure $\tilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ (i.e. $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0,\ A \in \mathcal{F}$), such that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$. (10 pts.)

3. Suppose that $X(t)$ satisfies the following Stochastic Differential Equation (SDE):

$$\mathrm{d}X(t) = 0.04X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),$$

and $Y(t)$ satisfies:

$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.1Y(t)\mathrm{d}W(t).$$

Parameters β, σ are positive constants and both processes are driven by the same Brownian Motion $W(t)$.
For a given process

$$Z(t) = 2\frac{X(t)}{Y(t)} - \lambda t,$$

with $\lambda \in \mathbb{R}^+$.

a. Find the SDE for $Z(t)$. (10 pts.)

b. For which values of β and λ is process $Z(t)$ a martingale? (10 pts.)

Z.O.Z. Remaining questions on the other side.

4. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDEs given by

$$\begin{aligned} \mathrm{d}S_1(t) &= 2S_1(t)\mathrm{d}W_1(t) + 3S_1(t)\mathrm{d}W_2(t), \\ \mathrm{d}S_2(t) &= S_2(t)\mathrm{d}t + S_2(t)\mathrm{d}W_1(t). \end{aligned}$$

(a) Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (10 pts.)

(b) Consider a finite time T (expiration date), and suppose the interest rate is a constant, i.e. $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t)) : 0 \leq t \leq T\}$. (10 pts.)

5. Assume we have a European call and a put option, with the same expiry date $T = 1/4$, i.e., exercise in three months, and strike price $K = 10$ Euro. The current share price is 11 Euro, assuming a constant interest rate $r = 6\%$. Determine an arbitrage opportunity if both options currently have the value $c(0) = p(0) = 2.5$ Euro. (10 pts.)

Please, make sure that your name is written down on each of the submitted solutions.