Utrecht University
Mathematical Institute

Answers re-Examation Introduction to Financial Mathematics, WISB373

Wednesday July 21th 2021, 15:15-18:15 o'clock **(3 hours examination)**

1. The stock price of company AB at time $t_0 = 0$ equals $S_0 = 10.0$€. A call option on this stock with exercise in 60 days and strike price $K = 10.5$€ costs $c(t_0) = 1$€. Prepare a table in which the stock price on the exercise date in the first row varies from 9.5€ to 12.5€. In the second row, put the profit/loss, in percentage, when buying the stock at t_0 and selling it at $t = T$; in the third row, the profit/loss at time $t = T$, in percentage, when buying the option at t_0. Compare the gains and losses in the option and stock returns.

	stock price at time $t = T$						
	9.5 €	10 €	10.5 €	11 €	11.5 €	12 €	12.5 €
profit stock	-5	0	5	10	15	20	25
profit option	-100	-100	-50	0	50	100	150

2. Show that, for a continuously, differentiable function $g(t)$, the process

$$X(t) = g(t)W(t) - \int_0^t g'(z)W(z)\mathrm{d}z,$$

is a martingale w.r.t. the natural filtration generated by the Brownian motion $W(t)$, where $g'(t)$ is the first derivative of $g(t)$, and subsequently show that

$$\mathbb{E}[\mathrm{e}^{2t}W(t)] = \mathbb{E}\left[\int_0^t 2\mathrm{e}^{2z}W(z)\mathrm{d}z\right].$$

Answer Ex. 2.: The corresponding SDE must not contain a drift term, if it represents a martingale process. The differential process of the integral equation is found to be,

$$\begin{aligned}\mathrm{d}X(t) &= d(g(t)W(t)) - g'(t)W(t)\mathrm{d}t \\ &= g'(t)W(t)\mathrm{d}t - g(t)\mathrm{d}W(t) - g'(t)W(t)\mathrm{d}t \\ &= -g(t)\mathrm{d}W(t),\end{aligned}$$

where the differentiation, denoted by d, is in the Itô sense. Hence, the process is a martingale, as we don't encounter any dt-term.

Express the term $e^{2t}W(t)$ in its integral form. as follows,

$$\begin{aligned} d(e^{2t}W(t)) &= 2e^{2t}W(t)\mathrm{d}t + e^{2t}\mathrm{d}W(t) \\ e^{2t}W(t) &= \int_t^0 (2e^{2u}W(u)\mathrm{d}u + e^{2u}\mathrm{d}W(u)),\end{aligned}$$

where $W(0) = 0$ is used in the second equation. Taking the expectation at both sides and using $\mathbb{E}[\int_0^t e^{2u}\mathrm{d}W(u)] = 0$, because infinitesimal increments of Brownian motion are governed by a normal distribution with zero mean. Therefore,

$$\mathbb{E}[e^{2t}W(t)] = \mathbb{E}\Big[\int_0^t 2e^{2u}W(u)\mathrm{d}t\Big].$$

3. A strangle, $St(t)$, is an option construction in which the investor takes a long position in a call and a put with different strike prices (K_1 for the call, and K_2 for the put, with $K_1 > K_2$), but with the same exercise date T and on the same stock S.

List the payout on the exercise date, distinguishing three different areas for the $St(T)$ share price. When would an investor buy a strangle in the case $K_2 << K_1$ and $K_2 < S_0 < K_1$?

Answer 3.: As it is clear from the diagram, there are three pay-off possibilities at the expiry date. In the left side of the diagram, the put option pay-off is positive whereas in the middle of the plot one does not gain anything from both options, and in the high asset value side of the plot one profits from the call option.

If the volatility of the asset prices is high and investment is unreliable, an investor can buy a strangle to prevent huge losses.

4. Suppose that $X(t)$ satisfies the following Stochastic Differential Equation (SDE):

$$\mathrm{d}X(t) = 0.02X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),$$

and $Y(t)$ satisfies:

$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.1Y(t)\mathrm{d}W(t).$$

Parameters β, σ are positive constants and both processes are driven by the same Brownian Motion $W(t)$.
For a given process

$$Z(t) = \frac{Y(t)}{X(t)} + (\lambda - \sigma)t,$$

with $\lambda \in \mathbb{R}$.

(a). Find the SDE for $Z(t)$.

(b). For what values of β, σ and λ is process $Z(t)$ a martingale?

Answer 4a. We have:

$$X(t) = e^{\sigma W(t) - \frac{\sigma^2}{2}t + 0.02t},$$
$$\mathrm{d}X(t) = 0.02X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t).$$

$$Y(t) = e^{0.1W(t) - \frac{0.01}{2}t + \beta t},$$
$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.1Y(t)\mathrm{d}W(t).$$

Using the expressions for $X(t)$ and $Y(t)$, we get,

$$Z(t) = e^{(0.1-\sigma)W(t) + (-0.02 - \frac{0.01}{2} + \beta + \frac{\sigma^2}{2})t} + (\lambda - \sigma)t.$$

Answer 4b. A martingale process does not contain a drift term. We have,

$$\begin{aligned}\mathrm{d}Z(t) &= (Z-(\lambda-\sigma)t)(-0.025+\beta+0.5\sigma^2)\mathrm{d}t \\ &\quad +(\lambda-\sigma)\mathrm{d}t+(Z-(\lambda-\sigma)t)(0.1-\sigma)\mathrm{d}W(t).\end{aligned}$$

With β and σ constant, and $\lambda \in \mathbb{R}^+$, the necessary conditions for a vanishing drift term are $\lambda = \sigma$ and

To check this result we employ the Itô's derivative rules for multivariate functions, i.e.,

$$\begin{aligned}\mathrm{d}Z(t) &= \left(\frac{\mathrm{d}Y(t)}{X(t)} - \frac{Y(t)\mathrm{d}X(t)}{X^2(t)} - \frac{\mathrm{d}Y(t)\mathrm{d}X(t)}{X^2(t)} + \frac{Y(t)\mathrm{d}X^2(t)}{X^3(t)}\right) = (\lambda-\sigma)\mathrm{d}t \\ &= (Z(t)+\lambda t)\big((0.04-\beta-0.1\sigma+0.01)\mathrm{d}t+(\sigma-0.1)\mathrm{d}W(t)\big)+(\lambda-\sigma)\mathrm{d}t,\end{aligned}$$

which yields the same constraints.

5. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the two processes $\{Z(t) : t \geq 0\}$ and $\{B(t) : t \geq 0\}$, defined by

$$Z(t) = 1 + \mathrm{e}^{-W_1(t)} \int_0^t \mathrm{e}^{W_1(u)} \mathrm{d}W_2(u),$$

and

$$B(t) = \int_0^t \frac{1}{\sqrt{1+Z^2(u)}} \mathrm{d}W_1(u) - \int_0^t \frac{Z(u)}{\sqrt{1+Z^2(u)}} \mathrm{d}W_2(u).$$

(a). Use Lévy's characterization to prove that process $\{B(t) : t \geq 0\}$ is a one-dimensional Brownian motion.

(b). Prove that the process $\{Z(t) : t \geq 0\}$ can be written as

$$Z(t) = 1 + W_2(t) - \int_0^t (Z(u)-1)\mathrm{d}W_1(u) + \frac{1}{2}\int_0^t (Z(u)-1)\mathrm{d}u.$$

Proof 5(a): We will use Levy's characterization of a Brownian motion. Clearly $B(0) = 0$ and since Itô integrals have continuous paths and are martingales, we see that $\{B(t) : t \geq 0\}$ has continuous paths and is a sum of two martingales; hence it is also a martingale. It remains to show that $[B, B](t) = t$. Note that

$$\mathrm{d}B(t) = \frac{1}{\sqrt{1+Z^2(t)}}\mathrm{d}W_1(t) - \frac{Z(t)}{\sqrt{1+Z^2(t)}}\mathrm{d}W_2(t),$$

thus

$$\mathrm{d}B(t)\mathrm{d}B(t) = \frac{1}{1+Z^2(t)}\mathrm{d}t + \frac{Z^2(t)}{1+Z^2(t)}\mathrm{d}t = \mathrm{d}t.$$

So, $[B, B](t) = t$ and by Lévy's characterization, $\{B(t) : t \geq 0\}$ is a one-dimensional Brownian motion.

Proof 5(b) Let $X(t) = \mathrm{e}^{-W_1(t)}$ and $Y(t) = \int_0^t \mathrm{e}^{W_1(u)}\mathrm{d}W_2(u)$, then $Z(t) = 1 + X(t)Y(t)$.

By definition, we have $\mathrm{d}Y(t) = \mathrm{e}^{W_1(t)}\mathrm{d}W_2(t)$. We will now derive the SDE for the process $\{X(t) : t \geq 0\}$, using the Itô-Doeblin formula applied to the function $f(x) = \mathrm{e}^{-x}$. We have $f_x(x) = -f(x)$ and $f_{xx}(x) = f(x)$, thus

$$\mathrm{d}X(t) = (W_1(t)) = -X(t)\mathrm{d}W_1(t) + \frac{1}{2}X(t)\mathrm{d}t,$$

that is

$$X(t) = 1 - \int_0^t X(u)\mathrm{d}W_1(u) + \frac{1}{2}\int_0^t X(u)\mathrm{d}u.$$

Since $W_1(t)$ and $W_2(t)$ are independent, $\mathrm{d}X(t)\mathrm{d}Y(t) = 0$. Applying the Itô product rule, we have,

$$\begin{aligned}
\mathrm{d}Z(t) = \mathrm{d}(1 + X(t)Y(t)) &= X(t)\mathrm{d}Y(t) + Y(t)\mathrm{d}X(t) + \mathrm{d}X(t)\mathrm{d}Y(t) \\
&= X(t)(\mathrm{e}^{W_1(t)}\mathrm{d}W_2(t)) + Y(t)(-X(t)\mathrm{d}W_1(t) + \frac{1}{2}X(t)\mathrm{d}t) \\
&= \mathrm{d}W_2(t) - X(t)Y(t)\mathrm{d}W_1(t) + \frac{1}{2}X(t)Y(t)\mathrm{d}t \\
&= \mathrm{d}W_2(t) - (Z(t) - 1)\mathrm{d}W_1(t) + \frac{1}{2}(Z(t) - 1)\mathrm{d}t.
\end{aligned}$$

Since $Z(0) = 1$, the above shows that

$$Z(t) = 1 + W_2(t) - \int_0^t (Z(u) - 1)\mathrm{d}W_1(u) + \frac{1}{2}\int_0^t (Z(u) - 1)\mathrm{d}u.$$

6. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by:

$$\begin{aligned}
\mathrm{d}S_1(t) &= \alpha S_1(t)\mathrm{d}W_1(t) + \beta S_1(t)\mathrm{d}W_2(t) \\
\mathrm{d}S_2(t) &= \gamma S_2(t)\mathrm{d}t + \sigma S_2(t)\mathrm{d}W_1(t),
\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

(a). Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process.

(b). Consider a finite expiration date T, and suppose the interest rate is constant, $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$.

Proof 6(a): We apply Itô's product rule, as follows,

$$\mathrm{d}(S_1(t)S_2(t)) = S_1(t)\mathrm{d}S_2(t) + S_2(t)\mathrm{d}S_1(t) + \mathrm{d}S_1(t)\mathrm{d}S_2(t).$$

Using $\mathrm{d}S_1(t) = \alpha S_1(t)\mathrm{d}W_1(t) + \beta S_1(t)\mathrm{d}W_2(t)$, $\mathrm{d}S_2(t) = \gamma S_2(t)\mathrm{d}t + \sigma S_2(t)\mathrm{d}W_1(t)$, and simplifying, we get

$$\mathrm{d}(S_1(t)S_2(t)) = (\gamma+\alpha\sigma)S_1(t)S_2(t)\mathrm{d}t+(\sigma+\alpha)S_1(t)S_2(t)\mathrm{d}W_1(t)+\beta S_1(t)S_2(t)\mathrm{d}W_2(t).$$

Equivalently,

$$\begin{aligned}
S_1(t)S_2(t) &= S_1(0)S_2(0) + \int_0^t (\gamma + \alpha\sigma)S_1(u)S_2(u)\mathrm{d}u \\
&\quad + \int_0^t (\sigma + \alpha)S_1(u)S_2(u)\mathrm{d}W_1(u) + \int_0^t \beta S_1(u)S_2(u)\mathrm{d}W_2(u).
\end{aligned}$$

Hence, $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô process.

Proof 6(b): Using the notation of the book, we have

$$\alpha_1 = 0, \sigma_{11} = \alpha, \sigma_{12} = \beta, \alpha_2 = \gamma, \sigma_{21} = \sigma, \sigma_{22} = 0.$$

The market price equations in this case form the system,

$$\begin{aligned} -r &= \alpha\theta_1(t) + \beta\theta_2(t) \\ \gamma - r &= \sigma\theta_1(t) \end{aligned}$$

Solving for $\theta_1(t), \theta_2(t)$, we get

$$\begin{aligned} \theta_1(t) &= \frac{\gamma - r}{\sigma} \\ \theta_2(t) &= -\frac{\sigma r + \alpha(\gamma - r)}{\sigma\beta} \end{aligned}$$

Setting

$$\begin{aligned} Z &= \exp\left\{-\int_0^T (\theta_1(t)\mathrm{d}W_1(t) + \theta_2(t)\mathrm{d}W_2(t)) - \frac{1}{2}\int_0^T \left(\theta_1^2(t) + \theta_2^2(t)\right)\mathrm{d}t\right\} \\ &= \exp\left\{-\frac{\gamma - r}{\sigma}W_1(T) + \frac{\sigma r + \alpha(\gamma - r)}{\sigma\beta}W_2(T) - \frac{1}{2}\left(\frac{(\gamma - r)^2}{\sigma^2} + \frac{(\sigma r + \alpha(\gamma - r))^2}{\sigma^2\beta^2}\right)T\right\} \end{aligned}$$

The risk-neutral measure is given by $\tilde{\mathbb{P}}(A) = \int_A Z\mathrm{d}\mathbb{P}$. To check this, we set

$$\tilde{W}_1(t) = \frac{\gamma - r}{\sigma}t + W_1(t)$$

and

$$\tilde{W}_2(t) = -\frac{\sigma r + \alpha(\gamma - r)}{\sigma\beta}t + W_2(t).$$

By the 2-dimensional Girsanov Theorem, the process $\{(\tilde{W}_1(t), \tilde{W}_2(t)) : 0 \leq t \leq T\}$ is a 2-dimensional Brownian motion under $\tilde{\mathbb{P}}$.
Rewriting $\mathrm{e}^{-rt}S_1(t), \mathrm{e}^{-rt}S_2(t)$ in terms of $\tilde{W}_1(t), \tilde{W}_2(t)$, we get after applying the Itô product rule

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{-rt}S_1(t)) &= \mathrm{e}^{-rt}S_1(t)(\alpha\mathrm{d}\tilde{W}_1(t) + \beta\mathrm{d}\tilde{W}_2(t)) \\ \mathrm{d}(\mathrm{e}^{-rt}S_2(t)) &= \mathrm{e}^{-rt}S_2(t)\sigma\mathrm{d}\tilde{W}_1(t), \end{aligned}$$

which shows that the discounted price processes are Itô integrals and hence martingales under $\tilde{\mathbb{P}}$.