Utrecht University
Mathematical Institute

Re-Examation for Introduction to Financial Mathematics, WISB373

Wednesday July 21th 2021, 15:15-18:15 o'clock **(3 hours examination)**

1. The stock price of company AB at time $t_0 = 0$ equals $S_0 = 10.0€$. A call option on this stock with exercise in 60 days and strike price $K = 10.5€$ costs $c(t_0) = 1€$. Prepare a table in which the stock price on the exercise date in the first row varies from 9.5€ to 12.5€. In the second row, put the profit/loss, in percentage, when buying the stock at t_0 and selling it at $t = T$; in the third row, the profit/loss at time $t = T$, in percentage, when buying the option at t_0. Compare the gains and losses in the option and stock returns.

	stock price at time $t = T$						
	9.5 €	10 €	10.5 €	11 €	11.5 €	12 €	12.5 €
profit stock							
profit option							

2. Show that, for a continuously, differentiable function $g(t)$, the process

$$X(t) = g(t)W(t) - \int_0^t g'(z)W(z)\mathrm{d}z,$$

is a martingale w.r.t. the natural filtration generated by the Brownian motion $W(t)$, where $g'(t)$ is the first derivative of $g(t)$, and subsequently show that

$$\mathbb{E}[\mathrm{e}^{2t}W(t)] = \mathbb{E}\left[\int_0^t 2\mathrm{e}^{2z}W(z)\mathrm{d}z\right].$$

3. A strangle, $St(t)$, is an option construction in which the investor takes a long position in a call and a put with different strike prices (K_1 for the call, and K_2 for the put, with $K_1 > K_2$), but with the same exercise date T and on the same stock S.

List the payout on the exercise date, distinguishing three different areas for the $St(T)$ share price. When would an investor buy a strangle in the case $K_2 << K_1$ and $K_2 < S_0 < K_1$?

Z.O.Z. Remaining questions on the other side.

4. Suppose that $X(t)$ satisfies the following Stochastic Differential Equation (SDE):

$$\mathrm{d}X(t) = 0.02X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),$$

and $Y(t)$ satisfies:

$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.1Y(t)\mathrm{d}W(t).$$

Parameters β, σ are positive constants and both processes are driven by the same Brownian Motion $W(t)$.
For a given process

$$Z(t) = \frac{Y(t)}{X(t)} + (\lambda - \sigma)t,$$

with $\lambda \in \mathbb{R}$.

(a). Find the SDE for $Z(t)$.

(b). For what values of β, σ and λ is process $Z(t)$ a martingale?

5. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the two processes $\{Z(t) : t \geq 0\}$ and $\{B(t) : t \geq 0\}$, defined by

$$Z(t) = 1 + \mathrm{e}^{-W_1(t)} \int_0^t \mathrm{e}^{W_1(u)} \mathrm{d}W_2(u),$$

and

$$B(t) = \int_0^t \frac{1}{\sqrt{1 + Z^2(u)}} \mathrm{d}W_1(u) - \int_0^t \frac{Z(u)}{\sqrt{1 + Z^2(u)}} \mathrm{d}W_2(u).$$

(a). Use Lévy's characterization to prove that process $\{B(t) : t \geq 0\}$ is a one-dimensional Brownian motion.

(b). Prove that the process $\{Z(t) : t \geq 0\}$ can be written as

$$Z(t) = 1 + W_2(t) - \int_0^t (Z(u) - 1)\mathrm{d}W_1(u) + \frac{1}{2}\int_0^t (Z(u) - 1)\mathrm{d}u.$$

6. Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by:

$$\begin{aligned} \mathrm{d}S_1(t) &= \alpha S_1(t)\mathrm{d}W_1(t) + \beta S_1(t)\mathrm{d}W_2(t) \\ \mathrm{d}S_2(t) &= \gamma S_2(t)\mathrm{d}t + \sigma S_2(t)\mathrm{d}W_1(t), \end{aligned}$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

(a). Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process.

(b). Consider a finite expiration date T, and suppose the interest rate is constant, $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\tilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$.

Please, make sure that your name is written down on each of the submitted solutions.