Utrecht University
Mathematical Institute

Mid-Term Exam for Introduction to Financial Mathematics, WISB373

Friday May 21th 2021, 13:15 - 15:15 **(2 hours examination)**

1. Flip a biased coin three times with $\mathbb{P}(H) = \frac{1}{4}$ and $\mathbb{P}(T) = \frac{3}{4}$. So our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\Omega = \{HHH; HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$\mathcal{F}$ is the power set of Ω, and

$$\begin{aligned}
\mathbb{P}(HHH) &= \frac{1}{64}, \\
\mathbb{P}(HHT) &= P(HTH) = P(THH) = \frac{3}{64}, \\
\mathbb{P}(HTT) &= P(THT) = P(TTH) = \frac{9}{64}, \\
\mathbb{P}(TTT) &= \frac{27}{64}.
\end{aligned}$$

Let $\mathcal{F}_1$ be the σ-algebra containing the information on the first coin flip, i.e., $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$, with $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$. Define a random variable X on Ω by

$$X = 16 \cdot \mathbb{1}_{\{HHH;HHT\}} + 8 \cdot \mathbb{1}_{\{HTH,HTT,THH,THT\}} + 4 \cdot \mathbb{1}_{\{TTH,TTT\}}.$$

a. Find an explicit expression for $\mathbb{E}[X|\mathcal{F}_1]$. (1 pt)

b. Define the price process S_0, S_1, S_2, S_3 on Ω by a tree, with $S_0 = 4$ and three coin tosses. Each time a head is tossed we have $S_i = 2S_{i-1}$, and each time a tail is obtained, we have $S_i = \frac{1}{2}S_{i-1}$.
Draw the corresponding tree, and show that $\sigma(S_2) \neq \mathcal{F}_2$. ($\mathcal{F}_2$ is the sigma algebra that contains the information about the first two coin flips.) (2 pt)

Proof: 1a. Since $\mathcal{F}_1$ is generated by the finite partition $\{A_H, A_T\}$, as we have seen in Homework 1,

$$\mathbb{E}[X|\mathcal{F}_1] = \frac{1}{\mathbb{P}(A_H)}\mathbb{1}_{A_H}\mathbb{E}[\mathbb{1}_{A_H}X] + \frac{1}{\mathbb{P}(A_T)}\mathbb{1}_{A_T}\mathbb{E}[\mathbb{1}_{A_T}X].$$

Now,

$$\mathbb{1}_{A_H}X = 16 \cdot \mathbb{1}_{\{HHH,HHT\}} + 8 \cdot \mathbb{1}_{\{HTH,HTT\}},$$

and

$$\mathbb{1}_{A_T}X = 8 \cdot \mathbb{1}_{\{THH,THT\}} + 4 \cdot \mathbb{1}_{\{TTH,TTT\}}.$$

Thus,

$$\frac{1}{\mathbb{P}(A_H)}\mathbb{1}_{A_H}\mathbb{E}[\mathbb{1}_{A_H}X] = 4(16 \cdot \mathbb{P}\{HHH; HHT\} + 8 \cdot \mathbb{P}\{HTH, HTT\})$$

and

$$\frac{1}{\mathbb{P}(A_T)}\mathbb{1}_{A_T}\mathbb{E}[\mathbb{1}_{A_T}X] = \frac{4}{3}(8 \cdot \mathbb{P}\{THH, THT\} + 4 \cdot \mathbb{P}\{TTH, TTT\}).$$

Therefore,
$$\mathbb{E}[X|\mathcal{F}_1] = 10\mathbb{1}_{A_H} + 5\mathbb{1}_{A_T}.$$

Proof: 1b. Flip a coin 3 times. Associated with this, we have the filtration $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$, that we have seen before:

$$\begin{aligned}
\mathcal{F}_1 &= \{\emptyset, \Omega, A_H, A_T\} \\
\mathcal{F}_2 &= \{\emptyset, \Omega, A_H, A_T, A_{HH}, A_{TT}, \ldots\} \\
\mathcal{F}_3 &= \mathcal{P}(\Omega)
\end{aligned}$$

Define S_0, S_1, S_2, S_3 on Ω by a tree:

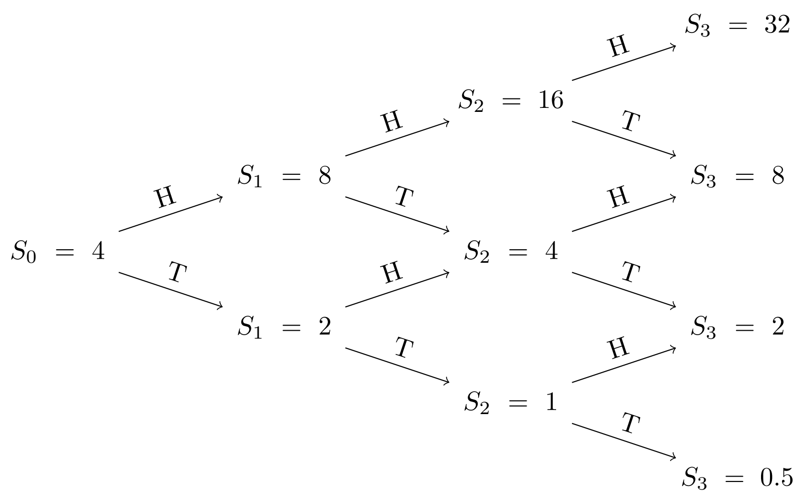

Figure 1: Tree for S_0, S_1, S_2, S_3 on Ω.

Consider S_2: To find $\sigma(S_2)$, it is enough to start with the sets $\{S_2 = 16\}, \{S_2 = 4\}, \{S_2 = 1\}$.
$\{S_2 = 16\} = A_{HH}, \{S_2 = 4\} = A_{HT} \bigcup A_{TH}, \{S_2 = 1\} = A_{TT}$.

So,

$$\begin{aligned}
\sigma(S_2) = \{&\emptyset, \Omega, A_{HH}, A_{HT} \bigcup A_{TH}, A_{TT}, \\
&A_{HH}^c = A_{HT} \bigcup A_{TH} \bigcup A_{TT}, \\
&A_{HH} \bigcup A_{TT} = (A_{HT} \bigcup A_{TH})^c, \\
&A_{TT}^c = A_{HH} \bigcup A_{HT} \bigcup A_{TH}\} \subset \mathcal{F}_2.
\end{aligned}$$

$A_{HT} \in \mathcal{F}_2$, however, $A_{HT} \notin \sigma(S_2)$.
Since $\mathcal{F}_2$ is finer, if we know in which element of $\mathcal{F}_2$ our outcome lies, then we know the value of S_2. The same, of course, is true for $\sigma(S_2)$.

2. Let $\{W(t) : t \geq 0\}$ be a Brownian motion, we define a process $\{X(t) : t \geq 0\}$ by

$$X(t) = \frac{1}{\sqrt{3}} W(3t).$$

a. Prove that $\{X(t) : t \geq 0\}$ is a Brownian motion. (1 pt)

b. Let $Y(t) = X^2(t) - 2\sqrt{c}t$ for some non-negative constant c and for all $t \geq 0$. For which value of c is the process $\{Y(t) : t \geq 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$, with $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$? (1 pt)

Proof: 2a. We check that the process $(X(t) : t \geq 0)$ satisfies all the properties of a Brownian motion. We have

(i)
$$X(0) = \frac{1}{\sqrt{3}} W(0) = 0.$$

(ii) Since $(W(t) : t \geq 0)$ has continuous paths and the function $t \to 3t$ is continuous and so is multiplication by $\frac{1}{\sqrt{3}}$, we see that the process $(X(t) : t \geq 0)$ also has continuous paths.

(iii) If $0 \leq u < v < s < t$, then clearly $0 \leq 3u < 3v < 3s < 3t$. Hence $W(3v) - W(3u)$ and $W(3t) - W(3s)$ are independent, implying that $X(v) - X(u)$ and $X(t) - X(s)$ are independent. Therefore, the process $(X(t) : t \geq 0)$ has independent increments.

(iv) If $s < t$, then $W(3t) - W(3s)$ is normally distributed with mean zero and variance $3t - 3s = 3(t - s)$. Hence $X(t) - X(s)$ is also normally distributed, with mean

$$\mathbb{E}[X(t) - X(s)] = \frac{1}{\sqrt{3}} \mathbb{E}[(W(3t) - W(3s)] = 0$$

and variance

$$\mathbb{V}\text{ar}(X(t) - X(s)) = \frac{1}{3} \mathbb{V}\text{ar}(W(3t) - W(3s)) = (t - s).$$

Therefore, $(X(t) : t \geq 0)$ is a Brownian Motion.

Proof: 2b. The underlying filtration is given by $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$. Now let $s < t$, and note that $Y(t) - Y(s)$ is independent of $\mathcal{F}(s)$, while $Y(s)$ is $\mathcal{F}(s)$-measurable. Hence,

$$\begin{aligned}
\mathbb{E}[Y(t)|\mathcal{F}(s)] &= \mathbb{E}[(Y(t) - Y(s)) + Y(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[Y(t) - Y(s)] + Y(s) \\
&= \mathbb{E}[X^2(t) - X^2(s)] - 2\sqrt{c}(t - s) + Y(s) \\
&= \mathbb{E}[\frac{1}{3} W^2(3t) - \frac{1}{3} W^2(3s)] - 2\sqrt{c}(t - s) + Y(s) \\
&= \frac{1}{3} 3t - \frac{1}{3} 3s - 2\sqrt{c}(t - s) + Y(s) \\
&= (1 - 2\sqrt{c})(t - s) + Y(s).
\end{aligned}$$

Hence, for the process $(Y(t) : t \geq 0)$ to be a martingale, we must have $c = 1/4$.

3. Suppose $\{W(t) : t \geq 0\}$ is a Brownian Motion, $\{\mathcal{F}(t) : t \geq 0\}$ is a filtration for $\{W(t) : t \geq 0\}$ and $\sigma > 0$. The Geometric Brownian Motion, GBM, $\{S(t) : 0 \leq t \leq T\}$ is defined by

$$S(t) = S(0) \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\}$$

with $\{W(t) : t \geq 0\}$ a BM. This process can be used to model certain asset prices, where parameter σ is the volatility.

The log-return on the interval $[t_i, t_{i+1}]$ is defined as,

$$\log\left(\frac{S(t_{i+1})}{S(t_i)}\right).$$

a. Show that, on the $0 \leq T_1 \leq T_2$, with the partition

$$T_1 = t_0 < t_1 < \ldots < t_m = T_2,$$

the quadratic variation of the log-returns gives us an estimate for the realized volatility in the time interval $[T_1, T_2]$. (2 pt)

b. Show that $S(t)$ is a martingale under the filtration $\mathcal{F}(t)$. (2 pt)

Proof: 3a. Let $0 \leq T_1 \leq T_2$ be given, and consider the partition

$$T_1 = t_0 < t_1 < \ldots < t_m = T_2.$$

(We think of $t_0, t_1, \ldots, t_m$ as the time points that we observe the prices.) The log-return on the interval $[t_i, t_{i+1}]$ is given by

$$\log\left(\frac{S(t_{i+1})}{S(t_i)}\right) = \sigma(W(t_{i+1}) - W(t_i)) + (\alpha - \frac{1}{2}\sigma^2)(t_{i+1} - t_i).$$

The realized volatility in $[T_1, T_2]$ is given by

$$\sum_{i=0}^{m-1}\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2,$$

which is the quadratic variation of the function $\log S(t)$ on the interval $[T_1, T_2]$ w.r.t. the partition.

$$\begin{aligned}\sum_{i=0}^{m-1}\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2 &= \sum_{i=0}^{m-1}\Big[\sigma^2(W(t_{i+1}) - W(t_i))^2 \\ &+ 2\sigma(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i) \\ &+ (\alpha - \frac{1}{2}\sigma^2)(t_{i+1} - t_i)^2\Big]\end{aligned}$$

If $||\mathcal{P}_m|| = ||\{t_0, \ldots, t_m\}||$ is small, then

$$\sum_{i=0}^{m-1}\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2 \approx \sigma^2(T_2 - T_1).$$

So

$$\sigma^2 \approx \frac{1}{(T_2 - T_1)}\sum_{i=0}^{m-1}\left(\log\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right)^2$$

is the realized volatility/averaged quadratic variation.

Proof: 3b.

$$Z(t) = \exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t\}$$

is $\mathcal{F}(t)$-measurable, i.e., $\{Z(t) : t \geq 0\}$ is adapted to the filtration. Let $s < t$ and consider:

$$\begin{aligned}
\mathbb{E}[Z(t)|\mathcal{F}(s)] &= \mathbb{E}\left[\exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t \pm \sigma W(s)\}\Big|\mathcal{F}(s)\right] \\
&= e^{-\frac{1}{2}\sigma^2 t} \cdot \mathbb{E}\left[\exp\{\sigma(W(t) - W(s)\} \cdot \exp\{\sigma W(s)\}|\mathcal{F}(s)\right] \\
&= e^{-\frac{1}{2}\sigma^2 t} e^{\sigma W(s)} \cdot \mathbb{E}[\exp\{\sigma(W(t) - W(s)\}|\mathcal{F}(s)] \\
&= e^{-\frac{1}{2}\sigma^2 t + \sigma W(s)} \cdot \mathbb{E}[e^{\{\sigma(W(t) - W(s)\}}] \\
&= e^{-\frac{1}{2}\sigma^2 t + \sigma W(s)} \cdot e^{\frac{1}{2}\sigma^2 (t-s)} \\
&= e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} = Z(s).
\end{aligned}$$

$\Rightarrow$ $\{Z(t) : t \geq 0\}$ is a martingale w.r.t the filtration $\{\mathcal{F}(t) : t \geq 0\}$.
We have used the fact that $W(t) - W(s) \sim N(0, t - s)$ and we have the moment generating function,

$$\varphi_{W(t)-W(s)}(u) = \mathbb{E}[e^{u(W(t)-W(s)}] = e^{\frac{1}{2}u^2(t-s)}.$$

In general, when $X \sim N(0, \sigma^2)$ then $\mathbb{E}[e^{uX}] = e^{\frac{1}{2}u^2\sigma^2}$.