

Utrecht University  
Mathematical Institute

**Mid-Term Exam for Introduction to Financial Mathematics,  
WISB373**

Friday May 21th 2021, 13:15 - 15:15 (**2 hours examination**)

1. Flip a biased coin three times with  $\mathbb{P}(H) = \frac{1}{4}$  and  $\mathbb{P}(T) = \frac{3}{4}$ . So our probability space is  $(\Omega, \mathcal{F}, \mathbb{P})$ , with

$$\Omega = \{HHH; HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$\mathcal{F}$  is the power set of  $\Omega$ , and

$$\begin{aligned}\mathbb{P}(HHH) &= \frac{1}{64}, \\ \mathbb{P}(HHT) &= P(HTH) = P(THH) = \frac{3}{64}, \\ \mathbb{P}(HTT) &= P(THT) = P(TTH) = \frac{9}{64}, \\ \mathbb{P}(TTT) &= \frac{27}{64}.\end{aligned}$$

Let  $\mathcal{F}_1$  be the  $\sigma$ -algebra containing the information on the first coin flip, i.e.,  $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$ , with  $A_H = \{HHH, HHT, HTH, HTT\}$  and  $A_T = \{THH, THT, TTH, TTT\}$ . Define a random variable  $X$  on  $\Omega$  by

$$X = 16 \cdot \mathbb{1}_{\{HHH; HHT\}} + 8 \cdot \mathbb{1}_{\{HTH, HTT, THH, THT\}} + 4 \cdot \mathbb{1}_{\{TTH, TTT\}}.$$

- a. Find an explicit expression for  $\mathbb{E}[X|\mathcal{F}_1]$ . (1pt)
  - b. Define the price process  $S_0, S_1, S_2, S_3$  on  $\Omega$  by a tree, with  $S_0 = 4$  and three coin tosses. Each time a head is tossed we have  $S_i = 2S_{i-1}$ , and each time a tail is obtained, we have  $S_i = \frac{1}{2}S_{i-1}$ . Draw the corresponding tree, and show that  $\sigma(S_2) \neq \mathcal{F}_2$ . ( $\mathcal{F}_2$  is the sigma algebra that contains the information about the first two coin flips.) (2pt)
2. Let  $\{W(t) : t \geq 0\}$  be a Brownian motion, we define a process  $\{X(t) : t \geq 0\}$  by

$$X(t) = \frac{1}{\sqrt{3}}W(3t).$$

- a. Prove that  $\{X(t) : t \geq 0\}$  is a Brownian motion. (1 pt)
- b. Let  $Y(t) = X^2(t) - 2\sqrt{c}t$  for some non-negative constant  $c$  and for all  $t \geq 0$ . For which value of  $c$  is the process  $\{Y(t) : t \geq 0\}$  a martingale with respect to the filtration  $\{\mathcal{F}(t) : t \geq 0\}$ , with  $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$ ? (1 pt)

**See other side for exercise 3.**

3. Suppose  $\{W(t) : t \geq 0\}$  is a Brownian Motion,  $\{\mathcal{F}(t) : t \geq 0\}$  is a filtration for  $\{W(t) : t \geq 0\}$  and  $\sigma > 0$ . The Geometric Brownian Motion, GBM,  $\{S(t) : 0 \leq t \leq T\}$  is defined by

$$S(t) = S(0) \exp \left\{ \sigma W(t) - \frac{1}{2} \sigma^2 t \right\}$$

with  $\{W(t) : t \geq 0\}$  a BM. This process can be used to model certain asset prices, where parameter  $\sigma$  is the volatility.

The log-return on the interval  $[t_i, t_{i+1}]$  is defined as,

$$\log \left( \frac{S(t_{i+1})}{S(t_i)} \right).$$

- a. Show that, on the  $0 \leq T_1 \leq T_2$ , with the partition

$$T_1 = t_0 < t_1 < \dots < t_m = T_2,$$

the quadratic variation of the log-returns gives us an estimate for the realized volatility in the time interval  $[T_1, T_2]$ . (2pt)

- b. Show that  $S(t)$  is a martingale under the filtration  $\mathcal{F}(t)$ . (2pt)

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Check whether your name and study number are on each of the pages with solutions!