Utrecht University
Block 3, 2020-21

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Name:________________________ ID #:__________ Signature:__________

WISB362 Stochastic Processes
Final Exam

This exam contains 14 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **Please, write your solutions in English!**
- **You have 3 hours to complete the exam**.
- **This is an open book exam.** You are allowed to use the textbook and/or lecture notes when working on it.
- **You are required to show your work on each problem on this exam.** All answers must be justified. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might receive partial credit.
- **If you use a theorem or proposition from class or the notes or the textbook you must indicate this** and explain why the theorem can be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Do not write in the table to the right.

Problem	Points	Score
1	20	
2	15	
3	20	
4	15	
5	15	
6	10	
Total:	95	

Good luck!

1. Answer the following questions. Justify each answer with a proof and/or a counterexample.

   (a) (5 points) Let T_1 and T_2 be stopping times for a Markov chain $(X_n)_{n\geq 0}$. Is $T := \max\{T_1, T_2\}$ a stopping time for $(X_n)_{n\geq 0}$?

(b) (5 points) Suppose that $X_0, X_1, \ldots$ are independent identically distributed random variables such that $\mathbf{P}(X_n = -1) = \mathbf{P}(X_n = 1) = \frac{1}{2}$, for each $n \geq 0$. Set $Y_0 = 0$ and $Y_n := X_n X_{n-1}$, for $n \geq 1$. Does the sequence $(Y_n)_{n\geq 0}$ form a martingale with respect to $(X_n)_{n\geq 0}$? If not, list all the conditions which are not satisfied.

(c) (5 points) Suppose $(N(t))_{t\geq 0}$ is a Poisson process with rate 2. Compute $\mathbf{P}(N(5) = 4|N(2) = 3)$ and $\mathbf{P}(N(2) = 3|N(5) = 4)$.

(d) (5 points) Let $a > 0$ be a real number. Let $(B(t))_{t\geq 0}$ be a standard Brownian motion. For any $t \geq 0$, define $X(t) := \frac{1}{\sqrt{a}} B(at)$. Show that $(X(t))_{t\geq 0}$ is also a standard Brownian motion.

2. Consider a Markov chain with $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\}$ and the following transition matrix:

$$P = \begin{array}{c|ccccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{2} & 0 & 0.6 & 0.4 & 0 & 0 & 0 & 0 \\ \mathbf{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{5} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{6} & 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0.8 \\ \mathbf{7} & 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0.2 \end{array}.$$

(a) (5 points) Find all closed irreducible sets in this Markov chain and classify all states as transient or recurrent. Justify your answer.

(b) (5 points) Find periods of states 3 and 5.

(c) (5 points) Compute a stationary distribution for this Markov chain. How many stationary distributions does it have?

3. An urn contains N balls - some red balls and some white balls. At each turn, a coin having probability $p \in (0,1)$ of landing heads is flipped and a ball is chosen uniformly at random from the urn. If heads appears, the ball is replaced by a red ball; if tails appears, the ball is replaced by a white ball (so that there are N balls in the urn after each turn). For $n \geq 0$, let X_n denote the number of red balls in the urn after the nth turn.

   (a) (5 points) Show that $(X_n)_{n\geq 0}$ is a Markov chain and find the transition matrix.

(b) (5 points) Is the Markov chain $(X_n)_{n\geq 0}$ irreducible? Is it reversible?

(c) (5 points) Suppose that $X_0 = j$, where $0 < j \leq N$. Find the probability that at each turn the urn has at least one red ball.

(d) (5 points) Find the limiting fraction of turns when the urn has 0 red balls.

4. Consider a population in which the offspring numbers of each individual are independent and identically distributed discrete random variables. More precisely, let X_n be the number of individuals in the nth generation, and let $Y_k^{(n+1)}$ be the offspring number of the kth individual in the nth generation. Then $Y_k^{(n)}$, $n, k \in \mathbb{N}$, are independent identically distributed with $range(Y_k^{(n)}) \subset \mathbb{N} \cup \{0\}$, $X_{n+1} = \sum_{k=1}^{X_n} Y_k^{(n+1)}$ for $n \geq 0$, and $X_0 = m$ for some $m \in \mathbb{N}$.

   (a) (5 points) Let $\mu = \mathbf{E}\left(Y_k^{(n)}\right)$. Find all values of μ, for which $(X_n)_{n\geq 0}$, is a martingale with respect to $(X_n)_{n\geq 0}$.

(b) (5 points) Assume now that the population goes extinct if it either dies out naturally (that is, if $X_n = 0$ for some $n \geq 1$), or due to lack of resources in the ecosystem, when the population reaches the bound of N individuals (that is, if $X_n \geq N$ for some $n \geq 1$).

Suppose that the distribution of $Y_k^{(n)}$ is given by $\mathbf{P}(Y_k^{(n)} = 0) = \mathbf{P}(Y_k^{(n)} = 2) = \frac{1}{4}$ and $\mathbf{P}(Y_k^{(n)} = 1) = \frac{1}{2}$. Show that the population eventually goes extinct with probability 1.

(c) (5 points) Consider the setup of part (b). Given $X_0 = m$, prove that the probability that the population goes extinct due to the lack of resources is at most $\frac{m}{N}$.

5. Alice works in a flower store. She arrives to work at time zero and spends all her time making flower bouquets. She makes bouquets according to a Poisson process with rate λ_A per time unit.

   (a) (5 points) Let T_k, $k \geq 1$, be the time at which Alice finished the kth bouquet. Suppose that Alice made only 1 bouquet by time 1. What is the conditional expectation of T_2 given this information?

   (b) (5 points) Suppose that each bouquet that Alice makes is (independently) a wedding bouquet with probability $\frac{1}{10}$ or a regular one with probability $\frac{9}{10}$. The prices of wedding bouquets are uniformly distributed on $[50, 100]$ and are independent of everything else (and of each other). Compute the expectation of the total price of wedding bouquets Alice made by time 3.

(c) (5 points) Alice's colleague Bob is late and arrives at the flower store only at time 1. He starts making the flower bouquets at time 1, and finishes them according to an independent Poisson process with rate λ_B. What is the probability mass function of the total number of bouquets made by Alice and Bob together during the interval $[0, 2]$? (Recall that Alice is making her bouquets starting time 0 and finishes them with rate λ_A.)

6. (10 points) The voters in a given town arrive at the place of voting according to a Poisson process with rate $\lambda = 100$ voters per hour. Assume that the voting place has infinitely many numbered voting booths, so that voters do not have to wait to vote. Assume further that after a voter arrives to the voting place, the amount of time (in hours) he needs to finish voting is uniformly distributed on $[0, 1/2]$ and independent of other voters. Upon arrival, a voter goes to the first voting booth if it is not occupied, or to the next free one, if the first voting booth is occupied. Find the limiting portion of time the first voting booth is occupied.