Utrecht University
Block 3, 2020-21

Instructor: P. Salanevich
Date: July 5, 2021

Name:____________________ ID #:__________ Signature:__________

WISB362 Stochastic Processes
Final Exam (Resit)

This exam contains 12 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **Please, write your solutions in English!**
- **You have 3 hours to complete the exam**.
- **This is an open book exam.** You are allowed to use the textbook and/or lecture notes when working on it.
- **You are required to show your work on each problem on this exam.** All answers must be justified. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might receive partial credit.
- **If you use a theorem or proposition from class or the notes or the textbook you must indicate this** and explain why the theorem can be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Do not write in the table to the right.

Problem	Points	Score
1	20	
2	15	
3	20	
4	10	
5	20	
6	10	
Total:	95	

Good luck!

1. Answer the following questions. Justify each answer with a proof and/or a counterexample.

   (a) (5 points) Let T_1 and T_2 be stopping times for a Markov chain $(X_n)_{n\geq 0}$. Is $T := \min\{T_1, T_2\}$ a stopping time for $(X_n)_{n\geq 0}$?

(b) (5 points) Suppose that $X_0, X_1, \ldots$ are independent identically distributed random variables such that $\mathbf{P}(X_n = -1) = \mathbf{P}(X_n = 1) = \frac{1}{2}$, for each $n \geq 0$. Set $Y_0 = 0$ and $Y_n := X_n + X_{n-1}$, for $n \geq 1$. Does the sequence $(Y_n)_{n\geq 0}$ form a martingale with respect to $(X_n)_{n\geq 0}$? If not, list all the conditions which are not satisfied.

(c) (5 points) Let $N(t)$ be a Poisson process with rate $\frac{\sqrt{5}-1}{2}$. Compute $\mathbf{P}(N(6) = 5|N(3) = 2)$ and $\mathbf{P}(N(3) = 2|N(6) = 5)$?

(d) (5 points) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Suppose that $0 < s < t$. Show that $\mathbf{E}[B(s)B(t)] = s$.

2. Consider a Markov chain with $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\}$ and the following transition matrix:

$$P = \begin{array}{c|ccccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{1} & 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \mathbf{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4} & 0 & 0.1 & 0 & 0.1 & 0 & 0.8 & 0 \\ \mathbf{5} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{6} & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ \mathbf{7} & 0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 \end{array}.$$

(a) (5 points) Find all closed irreducible sets in this Markov chain and classify all states as transient or recurrent. Justify your answer.

(b) (5 points) Find periods of states 3 and 5.

(c) (5 points) Compute a stationary distribution for this Markov chain. How many stationary distributions does it have?

3. A group of N friends are each either Republican or Democrat. In each round, 2 friends are randomly chosen (uniformly from the set of all possible pairs) and they discuss politics. If they are from the same party then they keep their original opinions, but if they disagree then one of them (at random) changes his opinion and favors the other party. Let X_n be the number of Democrats after n rounds.

   (a) (5 points) Observe that $(X_n)_{n\geq 0}$ is a Markov chain and find the transition matrix.

   (b) (5 points) Is the Markov chain irreducible? reversible?

(c) (5 points) Show that $(X_n)_{n\geq 0}$ is a martingale with respect to $(X_n)_{n\geq 0}$.

(d) (5 points) Given that $X_0 = j$, where $0 \leq j \leq N$, show that the probability that eventually all the friends are Democrats is equal to $\frac{j}{N}$.

4. (10 points) N balls are placed in K urns (in whatever way). Each minute, we choose one of the balls uniformly at random (that is, each ball is chosen with probability $1/N$) and place it in one of the urns also chosen uniformly at random (that is, each urn is chosen with probability $1/K$). Denote by X_n the number of balls in the first urn at time n. For each $n \in \mathbb{N}$, find real numbers a_n and b_n, such that $M_n = a_n X_n + b_n$ is a martingale with respect to $(X_n)_{n \geq 0}$.

5. A student receives emails and Facebook updates on his smartphone. The emails and Facebook updates are received independently of each other, and the gaps between two successive emails and two successive updates are independent and exponentially distributed with rates $\lambda_E = 10$ and $\lambda_F = 20$ per hour, respectively. The sizes of each email or update are independent of everything else and uniformly distributed on $[0, 1]$ Megabytes.

   (a) (5 points) What is the probability that the first email arrives before the first update?

   (b) (5 points) Write down the probability mass function of the total number of received emails and updates in one hour.

(c) (5 points) What is the expected amount of internet traffic spend, in total, over the first hour.

(d) (5 points) What is the limiting fraction of the traffic spend on receiving emails?

6. Assume that students arrive to Starbucks according to a Poisson process with rate λ and stay until Starbucks closes at a deterministic time T. Let us also suppose that the amount of money spend by a student is equal to the time that he spends in Starbucks. Let N be the total amount of students in the cafe at time T and W be the total revenue of Starbucks.

   (a) (5 points) For $k \geq 0$, compute $\mathbf{E}[W|N = k]$.

   (b) (5 points) Compute $\mathbf{E}[W]$.