Final exam, Mathematical Modelling (WISB357)

Monday, 1 Feb 2021, 15:15-18:15, online

- You may use your book and notes while working the problems.
- Scan your solutions and upload them to the Blackboard 'Assignment' within 40 minutes of the end of the exam (no later than 18:55).
- Sign the integrity statement below and submit your scanned signature with your solutions.
- Remain available online for 30 minutes (until 19:25) in MS Teams with your written solutions and a passport or photo ID card in case of random check.

`Solution.` `In small type-font letters.`

Integrity statement for online exams:
I hereby declare that I have prepared the solutions to this exam by myself, with no help from others, nor consulting any sources other than the course material, book and my own notes.
Name, student number, signature, date:

Problem 1. Assume traffic on a section of highway obeys the Greenshields law with maximum velocity $v_M = 1$ and maximum density $\rho_M = 1$.

(a) If traffic flows uniformly, for what speed v is the total number of cars per minute leaving the section of highway a maximum?

[2] Solution. The flux at x at time t is $J(x,t) = \rho(x,t)v(x,t)$. For the Greenshields law, $v = 1 - \rho$. Therefore, $J = \rho(1-\rho)$. The maximum occurs at $\rho = 1/2$, $v = 1/2$.

(b) For more general traffic situations, show that the density is constant along characteristics $\frac{dX}{dt} = c(\rho_0(x_0))$, where $X(0) = x_0$, $\rho_0(x) = \rho(x,0)$, $c(\rho) = 1 - 2\rho$.

[2] Solution. Suppose $\rho(x,t)$ is a solution of the traffic flow model $\rho_t + c(\rho)\rho_x = 0$, and $\rho(x,t)$ is differentiable in x and t. Along any curve $X(t)$ it holds that $\rho_t(X(t),t) + c(\rho(X(t),t))\rho_x(X(t),t) = 0$. On the other hand, it also holds that

$$\frac{d}{dt}\rho(X(t),t) = \rho_t(X(t),t) + \rho_x(X(t),t)\frac{dX}{dt}.$$

Consquently, $\rho(X(t),t)$ is constant along any curve $X(t)$ satisfying

$$\frac{dX}{dt} = c(\rho(X(t),t)).$$

However, in that case, $c(\rho(X(t),t)) = c(\rho(X(0),0))$ is constant, and the curve is a line, i.e., $c(\rho(X(t),t)) = c(\rho_0(x_0))$.

(c) Suppose the initial density is given by

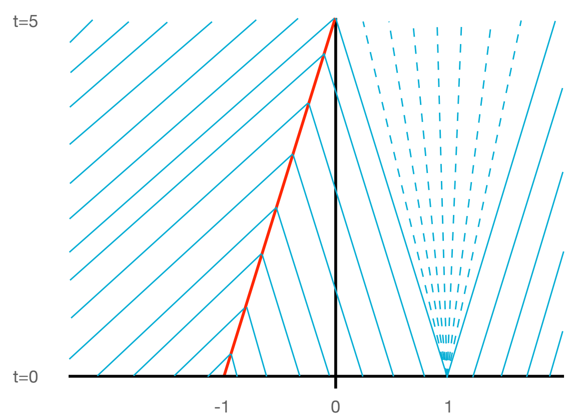
$$\rho_0(x) = \begin{cases} 1/5, & x < -1, \\ 3/5, & -1 \le x < 1, \\ 2/5, & 1 \le x. \end{cases}$$

Show that the solution supports a shock wave and an expansion fan, and determine the speed of the shock wave. Sketch the characteristics.

[3] Solution. For the Greenshields law, the wave speed is $c(\rho) = 1 - 2\rho$. A shock wave occurs whenever there is a discontinuous increase in density resulting in a discontinuous decrease in wave speed, in particular at $x_0 = -1$. The shock wave satisfies the Rankine-Hugoniot condition

$$\frac{dX_s}{dt} = \frac{1}{\rho_R - \rho_L} \int_{\rho_L}^{\rho_R} c(\rho)\, d\rho,$$

where ρ_L and ρ_R are the traffic densities to the left and right of the shock, respectively. In this case we find $\frac{dX_s}{dt} = 1 - (\rho_R + \rho_L) = \frac{1}{5}$, so the shock wave satisfies $X_s(t) = -1 + t/5$. An expansion fan occurs whenever there is a discontinuous decrease in density resulting in a discontinuous increase in wave speed, in particular at $x_0 = 1$ where we have $\rho_L = 3/5$, $c_L = c(\rho_L) = -1/5$ and $\rho_R = 2/5$, $c_R = c(\rho_R) = 1/5$. Here we have left characteristic $x_L(t) = x_0 + c_L t = 1 - t/5$ and right characteristic $x_R(t) = x_0 + c_R t = 1 + t/5$. The characteristics increase linearly in slope between these.



(d) At what time t_1 does the shock wave intersect the expansion fan? Give an expression for the solution $\rho(x,t)$ for $0 \le t \le t_1$, $-\infty < x < \infty$.

[3] Solution. The shock wave intersects the expansion fan for t such that $X_s(t) = x_L(t)$, i.e., $-1 + t_1/5 = 1 - t_1/5$. This happens at $t_1 = 5$, at which time $X_s = x_L = 0$. At a point (x,t) within the expansion fan, the density is

$$\rho = \rho_L + (\rho_R - \rho_L)\frac{x - x_L(t)}{x_R(t) - x_L(t)}$$

The solution up to time t_1 is

$$\rho(x,t) = \begin{cases} 1/5, & x < -1 + \frac{t}{5}, \\ 3/5, & -1 + \frac{t}{5} < x < 1 - \frac{t}{5}, \\ \frac{1}{2} - \frac{x-1}{2t}, & 1 - \frac{t}{5} \le x < 1 + \frac{t}{5}, \\ 2/5, & 1 + \frac{t}{5} \le x. \end{cases}$$

Problem 2. The momentum equation for a one-dimensional material with material coordinate $0 \le A \le \ell_0$ is given by

$$R_0(A)\frac{\partial^2 U}{\partial t^2}(A,t) = R_0(A)F(A,t) + \frac{\partial T}{\partial A}(A,t),$$

where R_0 is the density at time $t = 0$, $U(A,t)$ is the displacement, $F(A,t)$ is the net external body force, and $T(A,t)$ is the stress. Suppose that the external body force is independent of time: $F(A,t) = F(A)$ and that the stress depends only on the Lagrangian strain

$$T(A,t) = T\left(\frac{\partial U}{\partial A}\right).$$

(a) Prove that if $U(A,t)$ is a solution to the momentum equation, the total energy

$$\mathcal{E}(t) = \int_0^{\ell_0} \frac{1}{2} R_0 \left(\frac{\partial U}{\partial t}\right)^2 - R_0 F U + \phi\left(\frac{\partial U}{\partial A}\right) dA$$

is constant (i.e. $\frac{d\mathcal{E}}{dt} = 0$) along the solution, where ϕ is an anti-derivative (primitive) of T satisfying $\phi' = T$. Showing this will require an integration by parts. What assumptions do you need to make about the boundary conditions at $A = 0$ and $A = \ell_0$ for energy conservation to hold? (**Note:** the proof of part (a) is not needed for the rest of this problem. You may prefer to skip it for now and return to it later.)

[3] Solution. Computing

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \int_0^{\ell_0} R_0 U_t U_{tt} - R_0 F U_t + \phi'(U_A) U_{At}\, dA \\
&= \int_0^{\ell_0} R_0 U_t U_{tt} - R_0 F U_t - \frac{\partial}{\partial A}\left[\phi'(U_A)\right] U_t\, dA + \phi'(U_A) U_t \Big|_0^{\ell_0} \\
&= \int_0^{\ell_0} \left[R_0 U_{tt} - R_0 F - \frac{\partial}{\partial A} T(U_A) \right] U_t\, dA \\
&= 0,
\end{aligned}$$

where the term in square brackets is zero because of the momentum equation. The boundary terms arise from integration by parts. They are zero under conditions

$$(T(0,t) = 0 \text{ or } U_t(0,t) = 0) \quad \text{and} \quad (T(\ell_0,t) = 0 \text{ or } U_t(\ell_0,t) = 0).$$

The 'space elevator' is an idea dating from the end of the 19th century, in which a cable affixed to the earth is stretched so far into space that its centrifugal force counterbalances the pull of gravity. Subsequently, transport to orbital altitudes can be achieved by mechanical devices that 'climb' up and down the cable. In this problem, we develop formulas for the stress and deformation of such a cable. Assume the undeformed cable has length ℓ_0 and constant density R_0.

The gravitational and centrifugal forces are given in material coordinates by

$$F(A) = g\left(\frac{\bar{r}}{r(A)}\right)^2 - \omega^2 r(A),$$

where $r(A) = \bar{r} + A + U$, $\bar{r}$ is the radius of the Earth, $g > 0$ is the gravitational constant, and $\omega > 0$ the rotation rate of the Earth.

(b) Consider a steady state of the momentum equation in material coordinates. Assuming the displacement $U(A)$ is very small compared to $\bar{r}$, make the approximation $\bar{r} + A + U \approx \bar{r} + A$ and show that the equation of the steady state stress takes the form:

$$\frac{\partial T}{\partial A} = -R_0 g \left(\frac{\bar{r}}{\bar{r}+A}\right)^2 + R_0 \omega^2 (\bar{r} + A), \qquad T(\ell_0) = 0.$$

[2] Solution. For the steady state, all functions are independent of time. Hence $U_{tt} = 0$. The momentum equation becomes

$$\frac{\partial T}{\partial A} = -R_0(A)F(A).$$

Substituting the forces gives

$$\frac{\partial T}{\partial A} = -R_0(A)g\left(\frac{\bar{r}}{\bar{r}+A+U}\right)^2 + \omega^2 R_0(A)(\bar{r}+A+U).$$

Under the assumption $|U| \ll \bar{r}$, the above is approximated by

$$\frac{\partial T}{\partial A} = -R_0(A)g\left(\frac{\bar{r}}{\bar{r}+A}\right)^2 + \omega^2 R_0(A)(\bar{r}+A).$$

(c) By choosing appropriate scales for the time, space and mass, we can assume $\omega = \bar{r} = R_0 = 1$. (For instance, we measure time in days and distance in Earth radii.) Determine expressions for the solution $T(A)$ to the above problem and for the stress $T(0)$ at the Earth's surface. Is it conceivable to choose ℓ_0 such that $T(0) = 0$? What condition needs to hold on the (rescaled) gravitational constant g?

[3] Solution. Integrating the expression in part (b) from A to ℓ_0:

$$\begin{aligned} T(A) &= \int_A^{\ell_0} -g\left(\frac{1}{1+\alpha}\right)^2 + (1+\alpha)\,d\alpha \\ &= \left.\frac{g}{1+\alpha} + \alpha + \frac{\alpha^2}{2}\right|_A^{\ell_0} \\ &= \frac{g}{1+\ell_0} - \frac{g}{1+A} + (\ell_0 - A) + \frac{\ell_0^2 - A^2}{2}. \end{aligned}$$

It follows that

$$T(0) = \frac{g}{1+\ell_0} - g + \ell_0 + \frac{\ell_0^2}{2}.$$

The condition $T(0) = 0$ could hold if

$$2g\frac{\ell_0}{1+\ell_0} = \frac{\ell_0}{2}(2+\ell_0),$$

that is,

$$\ell_0^2 + 3\ell_0 + (2-2g) = 0 \quad\Longleftrightarrow\quad \ell_0 = -\frac{3}{2} \pm \frac{1}{2}\sqrt{1+8g},$$

which has a positive root if $g > 1$. One would need to check that the gravitational constant satisfies $g > 2$ under the chosen scaling.

(d) Assume the cable is linearly elastic with Young's modulus E. Determine an expression for the steady-state length ℓ of the cable.

[2] <u>Solution.</u> Let $T(A) = E\frac{\partial U}{\partial A}$. Integrating the expression above for $T(A)$ again yields

$$
\begin{aligned}
U(A) &= U(0) + \frac{1}{E}\int_0^A \frac{g}{1+\ell_0} - \frac{g}{1+\alpha} + (\ell_0 - \alpha) + \frac{\ell_0^2 - \alpha^2}{2}\, d\alpha \\
&= \frac{1}{E}\left[\left(\frac{g}{1+\ell_0} + \ell_0 + \frac{\ell_0^2}{2}\right)A - g\log(1+A) - \frac{A^2}{2} - \frac{A^3}{6}\right],
\end{aligned}
$$

under the boundary condition $U(0) = 0$ (since the cable is assumed affixed to the surface). The steady state length is $\ell = \ell_0 + U(\ell_0)$, or,

$$
\ell = \ell_0 + \frac{1}{E}\left[\frac{g\ell_0}{1+\ell_0} - g\log(1+\ell_0) + \frac{\ell_0^2}{2} + \frac{\ell_0^3}{3}\right].
$$