

Final exam, Mathematical Modelling (WISB357)

Monday, 1 Feb 2021, 15:15-18:15, online

-
- You may use your book and notes while working the problems.
 - Scan your solutions and upload them to the Blackboard ‘Assignment’ within 40 minutes of the end of the exam (no later than 18:55).
 - Sign the integrity statement below and submit your scanned signature with your solutions.
 - Remain available online for 30 minutes (until 19:25) in MS Teams with your written solutions and a passport or photo ID card in case of random check.
-

Integrity statement for online exams:

I hereby declare that I have prepared the solutions to this exam by myself, with no help from others, nor consulting any sources other than the course material, book and my own notes.

Name, student number, signature, date:

Problem 1. Assume traffic on a section of highway obeys the Greenshields law with maximum velocity $v_M = 1$ and maximum density $\rho_M = 1$.

- (a) [2pts] If traffic flows uniformly, for what speed v is the total number of cars per minute leaving the section of highway a maximum?
- (b) [2pts] For more general traffic situations, show that the density is constant along characteristics $\frac{dX}{dt} = c(\rho_0(x_0))$, where $X(0) = x_0$, $\rho_0(x) = \rho(x, 0)$, $c(\rho) = 1 - 2\rho$.
- (c) [3pts] Suppose the initial density is given by

$$\rho_0(x) = \begin{cases} 1/5, & x < -1, \\ 3/5, & -1 \leq x < 1, \\ 2/5, & 1 \leq x. \end{cases}$$

Show that the solution supports a shock wave and an expansion fan, and determine the speed of the shock wave. Sketch the characteristics.

- (d) [3pts] At what time t_1 does the shock wave intersect the expansion fan? Give an expression for the solution $\rho(x, t)$ for $0 \leq t \leq t_1$, $-\infty < x < \infty$.

Problem 2. The momentum equation for a one-dimensional material with material coordinate $0 \leq A \leq \ell_0$ is given by

$$R_0(A) \frac{\partial^2 U}{\partial t^2}(A, t) = R_0(A) F(A, t) + \frac{\partial T}{\partial A}(A, t),$$

where R_0 is the density at time $t = 0$, $U(A, t)$ is the displacement, $F(A, t)$ is the net external body force, and $T(A, t)$ is the stress. Suppose that the external body force is independent of time: $F(A, t) = F(A)$ and that the stress depends only on the Lagrangian strain

$$T(A, t) = T\left(\frac{\partial U}{\partial A}\right).$$

- (a) [3] Prove that if $U(A, t)$ is a solution to the momentum equation, the total energy

$$\mathcal{E}(t) = \int_0^{\ell_0} \frac{1}{2} R_0 \left(\frac{\partial U}{\partial t} \right)^2 - R_0 F U + \phi \left(\frac{\partial U}{\partial A} \right) dA$$

is constant (i.e. $\frac{d\mathcal{E}}{dt} = 0$) along the solution, where ϕ is an anti-derivative (primitive) of T satisfying $\phi' = T$. Showing this will require an integration by parts. What assumptions do you need to make about the boundary conditions at $A = 0$ and $A = \ell_0$ for energy conservation to hold? (**Note:** the proof of part (a) is not needed for the rest of this problem. You may prefer to skip it for now and return to it later.)

The ‘space elevator’ is an idea dating to the end of the 19th century, in which a cable affixed to the earth is stretched so far into space that its centrifugal force counterbalances the pull of gravity. Subsequently, transport to orbital altitudes can be achieved by mechanical devices that ‘climb’ up and down the cable. In this problem, we develop formulas for the stress and deformation of such a cable. Assume the undeformed cable has length ℓ_0 and constant density R_0 .

The gravitational and centrifugal forces are given in material coordinates by

$$F(A) = g \left(\frac{\bar{r}}{r(A)} \right)^2 - \omega^2 r(A),$$

where $r(A) = \bar{r} + A + U$, \bar{r} is the radius of the Earth, $g > 0$ is the gravitational constant, and $\omega > 0$ the rotation rate of the Earth.

- (b) [2] Consider a steady state of the momentum equation in material coordinates. Assuming the displacement $U(A)$ is very small compared to \bar{r} , make the approximation $\bar{r} + A + U \approx \bar{r} + A$ and show that the equation of the steady state stress takes the form:

$$\frac{\partial T}{\partial A} = -R_0 g \left(\frac{\bar{r}}{\bar{r} + A} \right)^2 + R_0 \omega^2 (\bar{r} + A), \quad T(\ell_0) = 0.$$

- (c) [3] By choosing appropriate scales for the time, space and mass, we can assume $\omega = \bar{r} = R_0 = 1$. (For instance, we measure time in days and distance in Earth radii.) Determine expressions for the solution $T(A)$ to the above problem and for the stress $T(0)$ at the Earth’s surface. Is it conceivable to choose ℓ_0 such that $T(0) = 0$? What condition needs to hold on the (rescaled) gravitational constant g ?
- (d) [2] Assume the cable is linearly elastic with Young’s modulus E . Determine an expression for the steady-state length ℓ of the cable.