WISB359 Mathematical Modelling

Midterm Exam, 10 December 2020, 9:00-10:45am

For this midterm exam you may use your book and notes. The exam will be graded on a scale of 20 points and is worth 50% of your score for the course.

1. The following partial differential equation represents the diffusion of a quantity with a point source:

$$\frac{\partial}{\partial t}u(x,t) = \mu\frac{\partial^2}{\partial x^2}u(x,t) - \alpha u(x,t) + \delta(x), \quad -\infty < x < \infty, \quad t > 0.$$

   Here, $\mu > 0$ is the diffusion parameter, $\alpha > 0$ is the dissipation parameter, and $\delta(x)$ is a generalized function known as the Dirac measure, which represents a source term. For the purpose of this problem, you may assume that the Fourier transform of the Dirac measure is given by:

$$D(k) = \frac{1}{\sqrt{2\pi}}, \quad -\infty < k < \infty, \qquad D = \mathcal{F}(\delta)$$

   (Strictly speaking there is an issue with this definition, but it will not cause problems here.)

   (a) [2pt] Let $U(k,t)$ denote the Fourier transform of $u(x,t)$. Derive the differential equation that $U(k,t)$ satisfies.

   (b) [2pt] Find any equilibria (=steady states) of your differential equation, and determine their stability.

   (c) [3pt] Suppose the initial condition is $u(x,0) = f(x)$, where the Fourier transform $F = \mathcal{F}(f)$ is well defined. Find the solution $U(k,t)$, $t > 0$.

   (d) [3pt] Determine the limit behavior ($t \to \infty$) of the solution $u(x,t)$. (In other words, is there a stable equilibrium for the system and what is that equilibrium, expressed as a function of x?)

2. The following model for a population of Aphids A and Ladybugs L in a rose garden takes into account both the saturation of the rose garden and the scale difference in lifespans of the Aphids (2 months) and Ladybugs (2-3 years):

$$\begin{aligned} A &\xrightarrow{r} 2A \\ A + A &\xrightarrow{s} A \\ A + L &\xrightarrow{p} (1+\epsilon)L \\ L &\xrightarrow{\delta} (1-\epsilon)L \end{aligned}$$

   Here, r is the reproduction rate of aphids, s the saturation parameter of the greenhouse, p the predation rate, δ the death rate of ladybugs (that is, use these constants as the reaction rates k_1, k_2, etc.). Furthermore, ϵ is a small parameter representing the ratio of lifespans of aphids/ladybugs. All parameters are assumed positive.

   (a) [1pt] Write down the associated system of differential equations for this model.

(b) [2pt] Introducing a rescaling of time $t = T\tau$ and choosing time scale $T = \epsilon^{-1}$, show that the system can be written in the form of a 'fast dynamics':

$$\epsilon \frac{dA}{d\tau} = f(A, L),$$
$$\frac{dL}{d\tau} = g(A, L),$$

where the functions f and g are independent of ϵ.

(c) [3pt] Suppose the initial populations are $A(0) = \alpha > 0$ and $L(0) = \lambda > 0$. Determine the equations that should be satisfied by a one-term expansion of the outer layer solution and show that these cannot satisfy the initial condition for A for arbitrary choice of λ and α.

(d) [3pt] Determine the rescaled differential equations that determines the inner layer solution for A (note: you do not have to solve these equations). Find the equilibria of these equations and explain how their stability depends on the parameters (r, s, p, δ) and the initial conditions λ and α.

(e) [1pt] For one equilibrium in part (d), both populations are nonzero. This equilibrium can be used to specify the number of aphids as a function of the number ladybugs (i.e. $A = A(L)$). On the time scale of the inner layer, the number of ladybugs is constant and the number of aphids rapidly approaches equilibrium. Use this relationship to define a reduced model consisting of a single differential equation for the number of ladybugs (i.e. $\dot{L} = q(L)$, where the function q is independent of A).

Statement for those taking the exam online:
I hereby declare that I have prepared the solutions to this exam by myself, with no help from others, nor consulting any sources other than the course material, book and my own notes.
Name, student number, signature, date: