6.2 Exam 2020/2021 (November 6th, 2020)

Note: The points below add up together to 13 points, so you do not have to solve everything to get the maximum mark of 10! Please motivate all your answers: do not just answer with ”yes” or ”no” but also provide arguments; do not just write down the final result, but also explain how you found it.

Exercise 6.6. Define

$$M = \{(x,y,z,t) \in \mathbb{R}^4 : xt = yz + 12\}.$$

(a) (*0.5 pt*) Show that M is an embedded submanifold of $\mathbb{R}^4$.
(b) (*0.5 pt*) Compute the tangent space of M at the point $p_0 = (4,0,0,3)$.
(c) (*0.5 pt*) Find a vector field $V \in \mathfrak{X}(M)$ with the property that its flow satisfies

$$\phi_V^{2s}(p_0) = (4\cos s, 4\sin s, -3\sin s, 3\cos s).$$

(d) (*1 pt*) Show that also

$$X^1 := \frac{1}{2}\left(y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} + t\cdot\frac{\partial}{\partial z} + z\cdot\frac{\partial}{\partial t}\right)$$

defines a vector field tangent to M and compute $X^2 := [X^1, V], X^3 := [X^1, X^2]$.
(e) (*1 pt*) Consider the spheres centred at the origin

$$S_r^3 = \{(x,y,z,t) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + t^2 = r^2\}$$

(one for each $r > 0$), and show that their intersections with M

$$M_r := M \cap S_r^3,$$

are embedded submanifolds of $\mathbb{R}^4$ (be aware that, depending on how you approach this, you may end up doing some very time consuming computations ...).
(f) (*0.5 pt*) When $r = 5$ (so that $p_0 \in M_r$) compute the tangent space of M_5 at p_0.
(g) (*0.5 pt*) Show that the following defines a submersion from M_5 to S^1:

$$f : M_5 \to S^1, \quad f(x,y,z,t) = (x-t, y+z).$$

(h) (*0.5 pt*) Show that M_5 is diffeomorphic to a torus.

Exercise 6.7. Return to the last exercise and, in (d) above, just assume that

$$X^1 := \frac{1}{2}\left(y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} + t\cdot\frac{\partial}{\partial z} + z\cdot\frac{\partial}{\partial t}\right),$$

$$X^2 := \frac{1}{2}\left(-x\cdot\frac{\partial}{\partial x} + y\cdot\frac{\partial}{\partial y} - z\cdot\frac{\partial}{\partial z} + t\cdot\frac{\partial}{\partial t}\right),$$

$$X^3 := \frac{1}{2}\left(-y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} - t\cdot\frac{\partial}{\partial z} + z\cdot\frac{\partial}{\partial t}\right)$$

and they satisfy

$$[X^1, X^2] = X^3, \quad [X^2, X^3] = -X^1, \quad [X^3, X^1] = -X^2$$

(you do not have to prove these identities). Show that:

(a) (*0.5 pt*) The 1-forms

$$\theta_1 = (-z\cdot dx + t\cdot dy + x\cdot dz - y\cdot dt)/12,$$

$$\theta_2 = (-t\cdot dx - z\cdot dy + y\cdot dz + x\cdot dt)/12,$$

$$\theta_3 = (\ z\cdot dx + t\cdot dy - x\cdot dz - y\cdot dt)/12$$

satisfy $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise.

(b) (*0.5 pt*) Deduce that X^1_p, X^2_p, X^3_p form a basis of T_pM for all $p \in M$.

(c) (*0.5 pt*) Deduce that two 2-forms $\eta, \xi \in \Omega^2(M)$ coincide if and only if

$$\eta(X^1,X^2) = \xi(X^1,X^2), \quad \eta(X^2,X^3) = \xi'(X^2,X^3), \quad \eta(X^3,X^1) = \xi(X^3,X^1).$$

(d) (*0.5 pt*) Show that $d\theta_1 = \theta_2 \wedge \theta_3, \quad d\theta_2 = \theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2$.

Exercise 6.8. (*1.5 pt*) Show that there exists a vector field $X \in \mathscr{X}(S^2)$ with flow given by

$$\phi_X^t(x,y,z) = \left(\frac{(1+x)e^t - (1-x)e^{-t}}{(1+x)e^t + (1-x)e^{-t}}, \ \frac{2y}{(1+x)e^t + (1-x)e^{-t}}, \ \frac{2z}{(1+x)e^t + (1-x)e^{-t}} \right).$$

Make sure you give all the details. Is the vector field that you found complete?

Exercise 6.9. Assume that M is a compact manifold, $f : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is projectable to $\frac{\partial}{\partial\theta}$, i.e.

$$(df)_p(V_p) = \left(\frac{\partial}{\partial\theta}\right)_{f(p)} \quad \text{for all } p \in M.$$

We also consider the pull-back via f of the canonical 1-form $d\theta \in \Omega^1(S^1)$,

$$\omega := f^*(d\theta) \in \Omega^1(M).$$

Show that:

(a) (*0.5 pt*) for each $\alpha \in \mathbb{R}$, the fiber

$$M_\alpha := \{p \in M : f(p) = e^{i\alpha}\}$$

is an embedded submanifold of M.

(b) (*1.5 pt*) ω is closed, $\omega(V) = 1$ and $\mathscr{L}_V(\omega) = 0$.

(c) (*1 pt*) for any $p \in M_\alpha$, $\phi_V^t(p) \in M_{\alpha+t}$ and the flow of V gives rise to diffeomorphisms

$$\phi_V^t|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad (\text{for all } t, \alpha \in \mathbb{R}).$$

Exercise 6.10. (*1.5 pt*) Consider the following copy of the torus in $\mathbb{R}^3$:

$$\mathbb{T}^2 := \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\}.$$

Choose an orientation on $\mathbb{T}^2$ and compute

$$\int_{\mathbb{T}^2} \omega, \quad \text{where } \omega = (x dy \wedge dz)|_{\mathbb{T}^2}.$$