6.1 Retake 2020/2021 (January 8-th, 2021)

Note: The points below add up together to 12 points, so you do not have to solve everything to get the maximum mark of 10! Please motivate all your answers: do not just answer with "yes" or "no" but also provide arguments; do not just write down the final result, but also explain how you found it.

Exercise 6.1. (*1 pt*) If $\omega \in \Omega^k(M)$ is a closed differential form on a manifold M show that the operation

$$\Omega^l(M) \to \Omega^{k+l}(M),\ \theta \mapsto \omega \wedge \theta$$

takes closed forms to closed forms, and exact forms to exact forms.

Exercise 6.2. (*1.5 pt*) Does there exists a vector field X on $\mathbb{R}^3$ with flow given by

$$\underset{X}{\phi^t}(x,y,z) = (x+t, y+tz, z)?$$

But a vector field Y with

$$\phi_Y^t(x,y,z) = (x+t, y+tz, z+t)?$$

Exercise 6.3. Consider the following differential 1-forms on S^2:

$$\theta_1 = x \cdot dy - y \cdot dx + z \cdot dz, \quad \theta_2 = x \cdot dy + y \cdot dx + z \cdot dz \quad \text{(restricted to } S^2\text{!)}$$

and the vector field $V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \in \mathfrak{X}(S^2)$.

(a) (*0.5 pt*) Which one of the 1-forms above is closed and which not?
(b) (*0.5 pt*) Which one of the 1-forms above is exact and which not?
(c) (*0.5 pt*) Is $\theta_1 \wedge \theta_2$ a volume form?
(d) (*0.5 pt*) Is $\theta_1 \wedge \theta_2$ exact?
(e) (*0.5 pt*) Compute $i_V(\theta_1 \wedge \theta_2)$.
(f) (*0.5 pt*) Show that $L_V(\theta_1) = 0$.
(g) (*1 pt*) Compute $(\phi_V^t)^* \theta_1$, where ϕ_V^t is the flow of V.

Exercise 6.4. Let $E : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, we denote by E_x, E_y and E_z the partial derivatives of E with respect to the variables x, y and z (each a smooth function on $\mathbb{R}^3$), and we construct the following three vector fields on $\mathbb{R}^3$:

$$\widetilde{U} = E_z \cdot \frac{\partial}{\partial y} - E_y \cdot \frac{\partial}{\partial z}, \quad \widetilde{V} = E_x \cdot \frac{\partial}{\partial z} - E_z \cdot \frac{\partial}{\partial x}, \quad \widetilde{W} = E_y \cdot \frac{\partial}{\partial x} - E_x \cdot \frac{\partial}{\partial y}.$$

We also consider

$$M_E := \{(x,y,z) \in \mathbb{R}^3 : E(x,y,z) = 0\}$$

and we assume that, at every point $p \in M_E$, at least one of the three vector fields above does not vanish. Show that:

(a) (*0.5 pt*) M_E is a 2-dimensional submanifold of $\mathbb{R}^3$.

(b) (*0.5 pt*) $\widetilde{U}, \widetilde{V}$ and $\widetilde{W}$ are tangent to M_E at each point $p \in M_E$. Hence they define three vector fields on M_E- and those will be denoted

$$U, V, W \in \mathfrak{X}(M_E).$$

(c) (*0.5 pt*) U, V and W span the tangent space T_pM_E at each $p \in M_E$.

(d) (*1 pt*) as a consequence of (c), at every point $p \in M$, the Lie bracket $[V,W]_p$ and the similar ones are linear combinations of U_p, V_p and W_p. Show that there are smooth functions f, g and h on M_E such that

$$[V,W] = f \cdot U + g \cdot V + h \cdot W,$$

and similarly for the other two Lie brackets.

(e) (*0.5 pt*) Consider the projection on the first two factors

$$\mathrm{pr}_{1,2} : M_E \to \mathbb{R}^2, \quad \mathrm{pr}_{1,2}(x,y,z) = (x,y).$$

Describe the points at which $\mathrm{pr}_{1,2}$ fails to be a submersion. And the same for the similar projections $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$.

(f) (*0.5 pt*) Show that, around each point $p \in M_E$, at least one of the projections $\mathrm{pr}_{1,2}$, $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$ can be used as a chart, i.e. there exists an open neighborhood U of p and a chart of type of type (U, χ), with χ being the restriction to U of one of those three projections.

Exercise 6.5. Consider the 3-dimensional projective space $\mathbb{P}^3$.

(a) (*0.5 pt*) Show that

$$p : (0,\pi) \times (0,\pi) \times (0,\pi) \to \mathbb{P}^3,$$

$$(\phi_1, \phi_2, \phi_3) \mapsto [\cos\phi_1 \ : \ \sin\phi_1 \cdot \cos\phi_2 \ : \ \sin\phi_1 \cdot \sin\phi_2 \cdot \cos\phi_3 \ : \ \sin\phi_1 \cdot \sin\phi_2 \cdot \sin\phi_3]$$

is a parametrization of $\mathbb{P}^3$, i.e. a diffeomorphism into an open $U \subset \mathbb{P}^3$.

(b) (*1.5 pt*) Assuming that you know already that $\mathbb{P}^3$ is orientable, and using any orientation that you prefer, compute

$$\int_{\mathbb{P}^3} f \cdot df_0 \wedge df_1 \wedge df_2,$$

where $f, f_0, f_1, f_2 : \mathbb{P}^3 \to \mathbb{R}$ are the functions which, for $(x_0, x_1, x_2, x_3) \in S^2$, send:

$$[x_0 : x_1 : x_2 : x_3] \xrightarrow{f_i} (x_i)^2, \quad [x_0 : x_1 : x_2 : x_3] \xrightarrow{f} x_0 \cdot x_1 \cdot x_2 \cdot x_3.$$

You can express the result in terms of $I_k = \int_0^\pi \sin^k \phi$ without further numerical computations (those would be obtained by further remarking that $I_k = \frac{k-1}{k} I_{k-2}$ and $I_0 = \pi$, $I_1 = 2$.)