Exam WISB326, June 28, 2021, 8:45-11:45
Solutions

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) Consider the subset of $\mathbb{P}^3(k)$ defined by

$$X = \{(1 : t : t^2 + t : t^2 - 2t) \in \mathbb{P}^3(k) : t \in k\} \cup \{(0 : 0 : 1 : 1)\}.$$

(a) (4 points) Show that $X \cap U_0$ is an affine algebraic set under an isomorphism $U_0 \cong \mathbb{A}^3(k)$.
(b) (8 points) Show that X is a projective algebraic set and that $(X_*)^* = X$.
(c) (8 points) Show that X_* is a curve. Is X a curve?

Solution

(a) $X \cap U_0 = \{(1 : t : t^2 + t : t^2 - 2t) \in \mathbb{P}^3(k) : t \in k\}$ is isomorphic to $\{(t, t^2 + t, t^2 - 2t) \in \mathbb{A}^3(k) : t \in k\} = V(x_3 + 2x_1 - x_1^2, x_2 - 3x_1 - x_3) \subseteq \mathbb{A}^3(k)$ under the isomorphism $U_0 \cong \mathbb{A}^3(k)$.

(b) Let $Y := X \cap U_0$. Then $Y^* = V(x_0x_3 + 2x_0x_1 - x_1^2, x_2 - 3x_1 - x_3) \subseteq \mathbb{P}^3(k)$ is a projective algebraic set and $X \subseteq Y^*$ as $(0 : 0 : 1 : 1) \in Y^*$. Moreover, $Y^* \cap U_0 = Y$ and $Y^* \cap V(x_0) = \{(0 : 0 : 1 : 1)\}$, thus $X = (Y^* \cap U_0) \cup (Y^* \cap V(x_0)) = Y^*$ is a projective algebraic set. Moreover, $X_* \cong Y$, thus $(X_*)^* = Y^* = X$.

(c) X_* is a curve if it is irreducible and of dimension 1. For irreducibility, $I(X_*) = (x_3 + 2x_1 - x_1^2, x_2 - 3x_1 - x_3)$, and $\Gamma(X_*) = k[x_1, x_2, x_3]/I(X_*) \cong k[x_1]$ is an integral domain. Hence $I(X_*)$ is a prime ideal and X_* is irreducible. For dimension, we either compute $k(X_*)$ isomorphic to $k(x_1)$ the function field of $\mathbb{A}^1(k)$, and hence of transcendence degree 1 over k. Alternatively, we see that the morphism $\mathbb{A}^1(k) \to X \cap U_0$ given by $t \mapsto (1 : t : t + t^2 : t^2)$ is an isomorphism with inverse $(x_0 : x_1 : x_2 : x_3) \mapsto x_1/x_0$, and hence X_* has dimension 1, as dimension is preserved under isomorphism. Since X_* is irreducible, $(X_*)^* = X$ is irreducible as well. Since X_* and X are birationally equivalent and dimension is preserved under birational equivalence, we have that X has dimension 1, and hence it is a curve.

(2) (12 points) Consider the irreducible algebraic set $X = V(x_1 + x_2^2 + x_3^3) \subseteq \mathbb{A}^3(k)$. Let

$$Y = X \cup \{(0, 0, 0), (1, 1, 1)\}.$$

For each of the following sets determine whether it is an open subset of Y and whether it is a dense subset of Y:

$$X, \quad Y \smallsetminus \{(0, 0, 0)\}, \quad \{(1, 1, 1)\}, \quad \{(0, 0, 0)\}.$$

Solution

Since $(1,1,1) \notin X$ and $(0,0,0) \in X$, we have $Y = X \cup \{(1,1,1)\}$. Since points are algebraic sets, then $X = Y \smallsetminus \{(1,1,1)\}$ and $Y \smallsetminus \{(0,0,0)\}$ are open subsets of Y by definition. Since $X \subseteq \mathbb{A}^3(k)$ is an algebraic set, then $Y \smallsetminus X = \{(1,1,1)\}$ is an open subset of Y. Since X and $\{(1,1,1)\}$ are both open subsets of Y and $X \cap \{(1,1,1)\} = \emptyset$, we conclude that neither of them is a dense subset of Y. Since $Y \smallsetminus \{(0,0,0)\}$ is an open subset of Y and $(Y \smallsetminus \{(0,0,0)\}) \cap \{(0,0,0)\} = \emptyset$, we conclude that $\{(0,0,0)\}$ is not a dense subset of Y. Since $k[x_1,x_2,x_3]/(x_1+x_2^2+x_3^3) \cong k[x_2,x_3]$ is an integral domain, then X is irreducible, and hence $X \smallsetminus \{(0,0,0)\}$ is dense in X as it is an open subset of an irreducible algebraic set. Thus $Y \smallsetminus \{(0,0,0)\} = X \smallsetminus \{(0,0,0)\} \cup \{(1,1,1)\}$ is dense in Y, because for every nonempty open subset U of Y, if $\{(1,1,1)\} \notin U$ we have $U \cap X \neq \emptyset$ and hence $U \cap (X \smallsetminus \{(0,0,0)\}) \neq \emptyset$. Finally, if $\{(0,0,0)\}$ was an open subset of Y, then $(Y \smallsetminus \{(0,0,0)\}) \cap \{(0,0,0)\} \neq \emptyset$, as $Y \smallsetminus \{(0,0,0)\}$ is dense in Y. This cannot happen, hence $\{(0,0,0)\}$ is not an open subset of Y.

(3) Consider the projective plane curves $X = V(f)$ and $Y = V(g)$ given by the polynomials

$$f = x_1^4 - 3x_1^2x_2^2 - 4x_0^2x_2^2, \qquad g = x_1^4 - 3x_0x_1^2x_2 - 4x_0^2x_2^2.$$

(a) (10 points) Compute the multiple points for f, and compute multiplicities and tangents at the multiple points for f.
(b) (8 points) Prove that $I((1:0:0), f \cap g) = 8$.
(c) (12 points) Compute all the points in the intersection $X \cap Y$, and for each point $x \in X \cap Y$ compute the intersection number $I(x, f \cap g)$.

Solution

(a) The multiple points for f can be computed as solutions of the system of equations $f = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$. We compute the partial derivatives of f:

$$\frac{\partial f}{\partial x_0} = -8x_0x_2^2, \qquad \frac{\partial f}{\partial x_1} = 4x_1^3 - 6x_1x_2^2, \qquad \frac{\partial f}{\partial x_2} = -6x_1^2x_2 - 8x_0^2x_2.$$

If $x_2 = 0$, then $f = 0$ implies $x_1 = 0$, and $(1:0:0)$ is a multiple point for f. If $x_2 \neq 0$, then $\frac{\partial f}{\partial x_0} = 0$ gives $x_0 = 0$, hence $\frac{\partial f}{\partial x_2} = 0$ gives $x_1 = 0$, and we get that $(0:0:1)$ is a multiple point of f. If we dehomogenize f with respect to x_0, we get $m_{(1:0:0)}(f) = m_{(0,0)}(x_1^4 - 3x_1^2x_2^2 - 4x_2^2) = 2$ and f has only one tangent line at $(1:0:0)$: $V(x_2)$. If we dehomogenize f with respect to x_2, we get $m_{(0:0:1)}(f) = m_{(0,0)}(x_1^4 - 3x_1^2 - 4x_0^2) = 2$ and f has two distinct tangent lines at $(0:0:1)$: $V(2x_0 + \sqrt{3}ix_1)$ and $V(2x_0 - \sqrt{3}ix_1)$, where $i \in k$ satisfies $i^2 = -1$.

(b) Let $P = (1:0:0)$. We dehomogenize g with respect to x_0 and see that $m_P(g) = m_{(0,0)}(x_1^4 - 3x_1^2x_2 - 4x_2^2) = 2$ and g has the same tangent line $V(x_2)$

as f at P.

$$I(P, f \cap g) = I((0,0), f_* \cap g_*) = I((0,0), f_* \cap (g_* - f_*))$$
$$= I((0,0), f_* \cap 3(x_1^2 x_2^2 - x_1^2 x_2))$$
$$= I((0,0), f_* \cap x_1^2) + I((0,0), f_* \cap x_2) + I((0,0), f_* \cap (3x_2 - 3))$$

By evaluating f_* at $x_1 = 0$ we get $I((0,0), f_* \cap x_1^2) = I((0,0), -4x_2^2 \cap x_1^2) = I((0,0), x_2^2 \cap x_1^2) = 4I((0,0), x_1 \cap x_1) = 4$ as $V(x_1)$ and $V(x_2)$ are nonsingular curves with distinct tangents at $(0,0)$. Similarly, $I((0,0), f_* \cap x_2) = I((0,0), x_1^4 \cap x_2) = 4I((0,0), x_1 \cap x_2) = 4$. Moreover, $I((0,0), f_* \cap (3x_2 - 3)) = 0$ as $(0,0) \notin V(3x_2 - 3)$. Thus $I(P, f \cap g) = 8$.

(c) We need to solve the system of equations $f = g = 0$. We solve the equivalent system of equations $f = g - f = 0$. Note that $g - f = 3x_1^2 x_2^2 - 3x_0 x_1^2 x_2 = 3x_1^2 x_2 (x_2 - x_0)$. If $x_1 = 0$ then $f = 0$ gives $x_0 x_2 = 0$, and hence the two points $(1 : 0 : 0)$ and $(0 : 0 : 1)$. If $x_2 = 0$ then $f = 0$ gives $x_1 = 0$ and we recover the point $(1 : 0 : 0)$. If $x_1 x_2 \neq 0$, then $g - f = 0$ gives $x_2 = x_0$ and $f = 0$ becomes $x_1^4 - 3x_2^2 - 4x_2^4 = 0$. Since $x_1^4 - 3x_2^2 - 4x_2^4 = (x_1^2 - 4x_2^2)(x_1^2 + x_2^2)$, we get $x_1 \in \{2x_2, -2x_2, ix_2, -ix_2\}$, where $i \in k$ satisfies $i^2 = -1$. Thus

$$X \cap Y = \{(1 : 0 : 0), (0 : 0 : 1), (1 : 2 : 1), (1, -2, 1), (1 : i : 1), (1 : -i : 1)\}.$$

Let $Q = (0 : 0 : 1)$. We dehomogenize g with respect to x_2 and see that $m_Q(g) = m_{(0,0)}(x_1^4 - 3x_0 x_1^2 - 4x_0^2) = 2$ and g has tangent line $V(x_0)$ at Q. Since f and g have no common tangent lines at Q we compute $I(Q, f \cap g) = m_Q(f) m_Q(g) = 4$. By Bezout's theorem we know that $\sum_{x \in X \cap Y} I(x, f \cap g) = \deg f \deg g = 16$. Hence,

$$\sum_{a \in \{\pm 2, \pm i\}} I((1 : a : 1), f \cap g) = \sum_{x \in X \cap Y} I(x, f \cap g) - I(P, f \cap g) - I(Q, f \cap g) = 16 - 12 = 4.$$

Since $I((1 : a : 1), f \cap g) \geq 1$ for each $a \in \{\pm 2, \pm i\}$, we conclude that $I((1 : a : 1), f \cap g) = 1$ for all $a \in \{\pm 2, \pm i\}$.

(4) Consider the morphisms:

$$\varphi : \mathbb{A}^2(k) \to \mathbb{A}^3(k), \quad (u, v) \mapsto (u, uv, v),$$
$$\psi : \mathbb{A}^3(k) \to \mathbb{P}^2(k), \quad (x, y, z) \mapsto (1 : x : y)$$

Let $X \subseteq \mathbb{A}^3(k)$ be the algebraic set defined by the irreducible polynomial

$$y^2 - x^2(x + 1).$$

Let $C = \varphi^{-1}(X) \setminus V(u)$.

(a) (12 points) Compute $I(\psi(X))$ and $I(C)$.

(b) (6 points) Let $C' = V(I(C)^*) \subseteq \mathbb{P}^2(k)$ and prove that C' is a nonsingular curve.

(c) (10 points) Let $C'' = V(I(\psi(X))) \subseteq \mathbb{P}^2(k)$, and let $\alpha : C' \dashrightarrow C''$ be the rational map induced by $\psi \circ \varphi|_C$. Prove that α is birational and a morphism. Is it an isomorphism?

Solution

(a) Let $g = x_0x_2^2 - x_1^2(x_1 + x_0) = f^*$. Note that $\psi(X) = \{(1 : x : y) \in \mathbb{P}^2(k) : f(x, y) = 0\} = V(g) \cap U_0$ and that $V(g) = \psi(X)^*$. Since $V(g)$ is the smallest projective algebraic set that contains $\psi(X)$, we know that $I(\psi(X)) = I(V(g))$. Since f is irreducible and $f^* = g$, we conclude that g is an irreducible polynomial and hence $I(V(g)) = (g)$. Note that

$$C = V(f(u, uv)) \smallsetminus V(u) = V(u^2(v^2 - u - 1)) \smallsetminus V(u) = V(v^2 - u - 1) \smallsetminus \{(0, 1), (0, -1)\}.$$

Since $k[u, v]/(v^2 - u - 1) \cong k[v]$ is an integral domain, $V(v^2 - u - 1)$ is an irreducible curve, hence $C \subseteq V(v^2 - u - 1)$ is a dense open subset, hence $V(v^2 - u - 1)$ is the smallest algebraic set that contains C, thus $I(C) = I(V(v^2 - u - 1)) = (v^2 - u - 1)$.

(b) $C' = V(v^2 - uw - w^2) \subseteq \mathbb{P}^2(k)$ is a curve as $g := v^2 - uw - w^2$ is an irreducible nonconstant polynomial. Moreover, $\frac{\partial g}{\partial u} = -w$, $\frac{\partial g}{\partial v} = 2v$, $\frac{\partial g}{\partial w} = -u - 2w$, so that $g = \frac{\partial g}{\partial u} = \frac{\partial g}{\partial v} = \frac{\partial g}{\partial w} = 0$ has no solutions in $\mathbb{P}^2(k)$, and C' is nonsingular.

(c) Note that C'' is a projective curve as it is a projective algebraic subset of $\mathbb{P}^2(k)$ defined by a noncostant irreducible polynomial. Since C' is nonsingular and C'' is projective the rational map α is a morphism. Note that $C = C' \smallsetminus V(w)$ and $\alpha(C) = C'' \smallsetminus V(x_0)$. Moreover,

$$\alpha|_{C \smallsetminus V(u)} : (C \smallsetminus V(u)) \to (\alpha(C) \smallsetminus V(x_1)), \quad (u : v : 1) \mapsto (1 : u : uv)$$

is injective with inverse

$$\beta : (\alpha(C) \smallsetminus V(x_1)) \to (C \smallsetminus V(u)), \quad (1 : x_1 : x_2) \mapsto \left(x_1 : \frac{x_2}{x_1} : 1\right),$$

and $C \smallsetminus V(u) = C' \smallsetminus V(uw)$ is an open subset of C', and $\alpha(C) \smallsetminus V(x_1) = C'' \smallsetminus V(x_0x_1)$ is an open subset of C''. Thus α is a birational map. However, α is not an isomorphism because it is not injective. Indeed $\alpha(0 : 1 : 1) = \alpha(0 : -1 : 1) = (1 : 0 : 0)$.

(5) (10 points) Let $\varphi : X \to Y$ be a birational morphism of curves. Let $P \in X$. Show that if $\varphi(P)$ is a simple point of Y, then P is a simple point of X.

Solution

Since $\varphi(P)$ is nonsingular $\mathcal{O}_{\varphi(P)}(Y)$ is a DVR. We want to prove that $\mathcal{O}_P(X)$ is a DVR. It suffices to prove that m_P is a principal ideal. Let $f \in m_P$. Let π be a generator of $m_{\varphi(P)}$. Then $f = u\pi^n$ for some $u \in \mathcal{O}_{\varphi(P)}(Y)$ and $n \in \mathbb{Z}$. If $n < 0$, then $1/f \in m_{\varphi(P)} \subseteq m_P$ which implies that $1 = f/f \in m_P$, a contradiction. If $n = 0$, then $f \in \mathcal{O}_{\varphi(P)}(Y)^\times \subseteq \mathcal{O}_{\varphi(P)}(Y)^\times$, which is a contradiction. Thus $n > 0$ and hence $f \in (\pi)$. We conclude that $m_p = (\pi)$.