Exam Introduction to Algebraic Varieties, WISB326
June 28, 2021, 8:45-11:45

EXAM PROBLEMS

- Let k be an algebraically closed field of characteristic 0.

(1) Consider the subset of $\mathbb{P}^3(k)$ defined by
$$X = \{(1 : t : t^2 + t : t^2 - 2t) \in \mathbb{P}^3(k) : t \in k\} \cup \{(0 : 0 : 1 : 1)\}.$$

(a) (4 points) Show that $X \cap U_0$ is an affine algebraic set under and isomorphism $U_0 \cong \mathbb{A}^3(k)$.

(b) (8 points) Show that X is a projective algebraic set and that $(X_*)^* = X$.

(c) (8 points) Show that X_* is a curve. Is X a curve?

(2) (12 points) Consider the irreducible algebraic set $X = V(x_1 + x_2^2 + x_3^3) \subseteq \mathbb{A}^3(k)$. Let
$$Y = X \cup \{(0, 0, 0), (1, 1, 1)\}.$$
For each of the following sets determine whether it is an open subset of Y and whether it is a dense subset of Y:
$$X, \quad Y \setminus \{(0, 0, 0)\}, \quad \{(1, 1, 1)\}, \quad \{(0, 0, 0)\}.$$

(3) Consider the projective plane curves $X = V(f)$ and $Y = V(g)$ given by the polynomials
$$f = x_1^4 - 3x_1^2x_2^2 - 4x_0^2x_2^2, \qquad g = x_1^4 - 3x_0x_1^2x_2 - 4x_0^2x_2^2.$$

(a) (10 points) Compute the multiple points for f, and compute multiplicities and tangents at the multiple points for f.

(b) (8 points) Prove that $I((1 : 0 : 0), f \cap g) = 8$.

(c) (12 points) Compute all the points in the intersection $X \cap Y$, and for each point $x \in X \cap Y$ compute the intersection number $I(x, f \cap g)$.

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(4) Consider the morphisms:

$$\varphi : \mathbb{A}^2(k) \to \mathbb{A}^3(k), \quad (u, v) \mapsto (u, uv, v),$$
$$\psi : \mathbb{A}^3(k) \to \mathbb{P}^2(k), \quad (x, y, z) \mapsto (1 : x : y)$$

Let $X \subseteq \mathbb{A}^3(k)$ be the algebraic set defined by the irreducible polynomial

$$y^2 - x^2(x + 1).$$

Let $C = \varphi^{-1}(X) \smallsetminus V(u)$.

(a) (12 points) Compute $I(\psi(X))$ and $I(C)$.

(b) (6 points) Let $C' = V(I(C)^*) \subseteq \mathbb{P}^2(k)$ and prove that C' is a nonsingular curve.

(c) (10 points) Let $C'' = V(I(\psi(X))) \subseteq \mathbb{P}^2(k)$, and let $\alpha : C' \dashrightarrow C''$ be the rational map induced by $\psi \circ \varphi|_C$. Prove that α is birational and a morphism. Is it an isomorphism?

(5) (10 points) Let $\varphi : X \to Y$ be a birational morphism of curves. Let $P \in X$. Show that if $\varphi(P)$ is a simple point of Y, then P is a simple point of X.

Success!