Retake exam WISB326, July 19, 2021, 11:30-14:30
Solutions

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) (12 points) Let $X = V((x_1+x_2)(2x_2-x_3), (x_1^2-x_2^2)(2x_2-x_1^2)) \subseteq \mathbb{A}^3(k)$. Determine the irreducible components of X.

Solution

Since $x_1^2 - x_2^2 = (x_1+x_2)(x_1-x_2)$, using the properties of $V(.)$ we have

$$\begin{aligned} X &= V(x_1+x_2) \cup V(2x_2-x_3, (x_1-x_2)(2x_2-x_1^2)) \\ &= V(x_1+x_2) \cup (V(2x_2-x_3) \cap V((x_1-x_2)(2x_2-x_1^2))) \\ &= V(x_1+x_2) \cup (V(2x_2-x_3) \cap V(x_1-x_2)) \cup (V(2x_2-x_3) \cap V(2x_2-x_1^2)) \\ &= V(x_1+x_2) \cup V(2x_2-x_3, x_1-x_2) \cup V(2x_2-x_3, 2x_2-x_1^2). \end{aligned}$$

Now, x_1+x_2 is an irreducible polynomial, $(2x_2-x_3, x_1-x_2)$ is a prime ideal as it is generated by linear polynomials, and $k[x_1,x_2,x_3]/(2x_2-x_3, 2x_2-x_1^2) \cong k[x_1]$ is an integral domain, hence $X_1 := V(x_1+x_2)$, $X_2 := V(2x_2-x_3, x_1-x_2)$ and $X_3 := V(2x_2-x_3, 2x_2-x_1^2)$ are irreducible. We observe that $X_1 \cap X_2 = \{(0,0,0)\}$ is a proper subset of both X_1 and X_2, hence $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$. Similarly, $X_1 \not\subseteq X_3$ and $X_3 \not\subseteq X_1$ as $X_1 \cap X_3 = \{(0,0,0), (-2,2,4)\}$ is a proper subset of both X_1 and X_3. Finally, $X_2 \cap X_3 = \{(0,0,0), (2,2,4)\}$ is a proper subset of both X_2 and X_3, hence $X_2 \not\subseteq X_3$ and $X_3 \not\subseteq X_2$. Thus X_1, X_2, X_3 are the irreducible components of X.

(2) (12 points) Let $X = V(x_1 - x_2) \subseteq \mathbb{A}^3(k)$. For each of the following sets determine whether it is an open subset of X and whether it is a closed subset of X:

$$(X \cap V(x_1)) \smallsetminus \{(1,1,1)\}, \qquad (X \cap V(x_1^2 - x_2^2)) \smallsetminus \{(1,1,0)\},$$
$$(X \cap V(x_1+x_2, x_3^3 - x_3)) \smallsetminus \{(0,0,0)\}.$$

Solution

Since $(1,1,1) \notin V(x_1)$, we see that $(X \cap V(x_1)) \smallsetminus \{(1,1,1)\} = X \cap V(x_1)$ is an intersection of two algebraic sets, hence it is an algebraic set in $\mathbb{A}^2(k)$, and hence, a closed subset of X. Since X is a line, it is irreducible, and hence every nonempty open subset is dense in X. We see that $(0,0,0) \in X \cap V(x_1)$, thus $X \cap V(x_1)$ is nonempty. Since $X \cap V(x_1)$ is closed in X and $(1,1,1) \notin X \cap V(x_1)$, the complement of $X \cap V(x_1)$ is a nonempty open subset of X, and it doesn't intersect $X \cap V(x_1)$. Thus $X \cap V(x_1)$ is not dense in X, and hence it is not an open subset of X.

We observe that $X \subseteq V(x_1^2 - x_2^2)$, as $V(x_1^2 - x_2^2) = V(x_1 - x_2) \cup V(x_1 + x_2)$. Thus $(X \cap V(x_1^2 - x_2^2)) \smallsetminus \{(1,1,0)\} = X \smallsetminus \{(1,1,0)\}$ is an open subset of X because its complement $\{(1,1,0)\}$, which is a point, is an algebraic set. Since

$(0,0,0) \in X \setminus \{(1,1,0)\}$, we have that $X \setminus \{(1,1,0)\}$ is a nonempty open subset of X, and hence dense in X. Thus X is the smallest algebraic set that contains $X \setminus \{(1,1,0)\}$, hence $X \setminus \{(1,1,0)\}$ is not an algebraic set, and in particular, it is not a closed subset of X.

We observe that $X \cap V(x_1 + x_2, x_3^3 - x_3) = \{(0,0,0),(0,0,1),(0,0,-1)\}$, and hence $(X \cap V(x_1+x_2, x_3^3 - x_3)) \setminus \{(0,0,0)\} = \{(0,0,1),(0,0,-1)\}$ is a finite set of points, thus and algebraic set and hence a closed subset of X. If $\{(0,0,1),(0,0,-1)\}$ was an open subset of X, it would be dense, because X is irreducible. But this is not possible because the complement of $\{(0,0,1),(0,0,-1)\}$ in X is a nonempty open subset of X that does not intersect $\{(0,0,1),(0,0,-1)\}$. Thus $(X \cap V(x_1 + x_2, x_3^3 - x_3)) \setminus \{(0,0,0)\}$ is not open in X.

(3) Consider the subset of $\mathbb{P}^2(k)$ defined by

$$X = \{(1 : t^{-1} : t) \in \mathbb{P}^2(k) : t \in k \setminus \{0\}\}.$$

(a) (6 points) Show that $X \cap U_0$ is an irreducible affine plane curve under an isomorphism $U_0 \cong \mathbb{A}^2(k)$.
(b) (6 points) Show that X is not a projective algebraic set.
(c) (6 points) Let $Y \subseteq \mathbb{P}^2(k)$ be a projective algebraic set that contains X and such that X is dense in Y. Prove that $I(X) = I(Y)$.

Solution

(a) Observe that $X \subseteq U_0$, and hence $X \cap U_0 = X$. Let $\varphi_0 : \mathbb{A}^2(k) \to U_0$ be the isomorphism given by $\varphi_0(x_1, x_2) = (1 : x_1 : x_2)$. Then $Z := \varphi_0^{-1}(X \cap U_0) = \{(t^{-1}, t) \in \mathbb{A}^2(k) : t \in k\} = V(x_1x_2 - 1)$ is an affine algebraic set. Since $f = x_1x_2 - 1$ is a nonconstant polynomial, Z is an affine plane curve. Since $k[x_1, x_2]/(f) \subseteq k(x_1)$, we deduce that it is an integral domain. Hence, the ideal (f) is prime and f is an irreducible polynomial. Thus Z is irreducible.
(b) If X is a projective algebraic set then it is a projective plane curve, because it is irreducible and of dimension 1 by part (a). Thus by Bezout's theorem we have $X \cap V(x_0) \neq \emptyset$, which is a contradiction as $X \subseteq U_0 = \mathbb{P}^2(k) \setminus V(x_0)$.
(c) Since $X \subseteq Y$ we have $I(Y) \subseteq I(X)$. If $I(Y) \neq I(X)$, then there is a homogeneous polynomial f in $I(X) \setminus I(Y)$. Thus $U := Y \setminus V(f)$ is a nonempty open subset of Y, and $X \cap U = \emptyset$, which contradicts the assumption that X is dense in X.

(4) Consider the projective plane curve $X = V(f)$ given by the polynomial

$$f = x_0^6 + (x_1 - x_2)^2(x_1^2x_2^2 - 2x_1x_2^3).$$

(a) (4 points) Compute a change of coordinates $\varphi : \mathbb{P}^2(k) \to \mathbb{P}^2(k)$ such that $\varphi(0:0:1) = (0:1:1)$, $\varphi(0:1:0) = (0:1:0)$ and $\varphi(V(x_0)) = V(x_0)$.
(b) (12 points) Compute the multiple points for f and compute multiplicities and tangent lines at the multiple points for f.
(c) (10 points) Compute the intersection number $I((0:1:1), f \cap g)$ for

$$g = x_0^3 - (x_1 - x_2)^2(x_1 + x_2).$$

Solution

(a) The morphism $\varphi : \mathbb{P}^2(k) \to \mathbb{P}^2(k)$ given by $\varphi(x_0 : x_1 : x_2) = (x_0 : x_1 + x_2 : x_2)$ is a change of coordinates because it is an isomorphism (with inverse

$\varphi^{-1}(x_0 : x_1 : x_2) = (x_0 : x_1 - x_2 : x_2))$, and it satisfies the required conditions as $x_0 \circ \varphi = x_0$.

(b) The multiple points for f can be computed as solutions of the linear system of equations $f = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$. We compute the partial derivatives of f:

$$\frac{\partial f}{\partial x_0} = 6x_0^5,$$
$$\frac{\partial f}{\partial x_1} = 2(x_1 - x_2)(x_1^2x_2^2 - 2x_1x_2^3) + (x_1 - x_2)^2(2x_1x_2^2 - 2x_2^3)$$
$$\frac{\partial f}{\partial x_2} = -2(x_1 - x_2)(x_1^2x_2^2 - 2x_1x_2^3) + (x_1 - x_2)^2(2x_1^2x_2 - 6x_1x_2^2).$$

We see that $\frac{\partial f}{\partial x_0} = 0$ gives $x_0 = 0$, and hence, instead of $f = 0$ we can solve

$$(x_1 - x_2)^2x_1x_2^2(x_1 - 2x_2) = 0.$$

If $x_1 = x_2$ we get the multiple point $(0 : 1 : 1)$. If $x_1 = 0$, $\frac{\partial f}{\partial x_1} = 0$ gives $x_2 = 0$ and hence no solution in $\mathbb{P}^2(k)$. If $x_2 = 0$ we get the multiple point $(0 : 1 : 0)$. If we dehomogenize f with respect to x_1, we get $m_{(0:1:0)}(f) = m_{(0,0)}(x_0^6 + (1 - x_2)^2(x_2^2 - 2x_2^3)) = 2$ and f has only one tangent line at $(0 : 1 : 0)$: $V(x_2)$. For the multiplicity of f at $(0 : 1 : 1)$ we first compute

$$f \circ \varphi = x_0^6 + x_1^2x_2^2((x_1 + x_2)^2 - 2(x_1 + x_2)x_2) = x_0^6 + x_1^2x_2^2(x_1^2 - x_2^2).$$

If we dehomogenize $f \circ \varphi$ at x_2, we get

$$m_{(0:1:1)}(f) = m_{(0:0:1)}(f \circ \varphi) = m_{(0,0)}(x_0^2 + x_1^2(x_1^2 - 1)) = 2,$$

and $f \circ \varphi$ has only one tangent line at $(0 : 0 : 1)$: $V(x_1)$. Thus f has only one tangent line at $(0 : 1 : 1)$: $V(x_1 - x_2)$.

(c) Since intersection numbers are invariant under changes of variables, we have $I((0 : 1 : 1), f \cap g) = I((0 : 0 : 1), (f \circ \varphi) \cap (g \circ \varphi))$, where $f \circ \varphi = x_0^6 + x_1^2x_2^2(x_1^2 - x_2^2)$ and $g \circ \varphi = x_0^3 - x_1^2(x_1 + 2x_2)$. By dehomogenizing with respect to x_2 we get

$$I((0 : 0 : 1), (f \circ \varphi) \cap (g \circ \varphi)) = I((0, 0), f' \cap g'),$$

where $f' = (f \circ \varphi)_* = x_0^6 + x_1^2(x_1^2 - 1)$ and $g' = (g \circ \varphi)_* = x_0^3 - x_1^2(x_1 + 2)$. Moreover,

$$\begin{aligned} I((0, 0), f' \cap g') &= I((0, 0), (f' - x_0^3g') \cap g') \\ &= I((0, 0), (x_1^2(x_1^2 - 2x_0^3 + x_0^3x_1 - 1) \cap (x_0^3 - x_1^2(x_1 + 2)) \\ &= I((0, 0), x_1^2 \cap (x_0^3 - x_1^2(x_1 + 2)) = I((0, 0), x_1^2 \cap x_0^3) = 6. \end{aligned}$$

(5) (10 points) Let $f \in k[x_0, x_1, x_2]$ be an irreducible homogeneous polynomial of degree 6. Let $Q_1, Q_2, Q_3 \in V(f) \cap V(x_0) \subseteq \mathbb{P}^2(k)$ be three distinct points, and assume that Q_1, Q_2, Q_3 are all multiple points for f. Prove that $I(Q_i, f \cap x_0) = 2$ for all $i \in \{1, 2, 3\}$, and that $V(x_0)$ is not tangent to $V(f)$ at Q_1.

Solution

Since $V(x_0)$ is a line and hence nonsingular, we know that $m_{Q_i}(x_0) = 1$ for all $i \in \{1, 2, 3\}$. Since Q_1, Q_2, Q_3 are multiple points for f we have $m_{Q_i}(f) \geq 2$ for all $i \in \{1, 2, 3\}$. Hence, we have $I(Q_i, f \cap x_0) \geq m_{Q_i}(f)m_{Q_i}(x_0) \geq 2$

for all $i \in \{1,2,3\}$. By Bezout's theorem we know that $\sum_{i=1}^{3} I(Q_i, f \cap x_0) \leq \deg f \deg x_0 = 6$, thus we conclude that $I(Q_i, f \cap x_0) = 2$ for all $i \in \{1,2,3\}$. Since $2 = m_{Q_1}(f) m_{Q_1}(x_0) \leq I(Q_i, f \cap x_0) = 2$, we have $m_{Q_1}(f) m_{Q_1}(x_0) = I(Q_i, f \cap x_0)$ and hence $V(x_0)$ and $V(f)$ cannot have common tangents at Q_1. Since the tangent line to $V(x_0)$ at Q_1 is $V(x_0)$, we conclude that $V(x_0)$ is not tangent to $V(f)$.

(6) Let $X = V(x_2^2 - x_1^3) \subseteq \mathbb{A}^2(k)$ and consider the morphism

$$\varphi : X \to \mathbb{P}^1(k), \quad (x_1, x_2) \mapsto (1 : x_1).$$

(a) (4 points) Prove that φ is a dominant rational map.

(b) (8 points) Prove that φ is not birational.

Solution

(a) Since k is algebraically closed, for every $x_1 \in k$ there is $x_2 \in k$ such that $x_2^2 = x_1^3$. Thus φ is surjective onto U_0. Since $\mathbb{P}^1(k)$ is irreducible and U_0 is open in $\mathbb{P}^1(k)$, then U_0 is dense in $\mathbb{P}^1(k)$ and hence φ is dominant. Every morphism is a rational map, thus φ is a rational map.

(b) If φ is a birational map, then there are open subsets $U \subseteq X$ and $V \subseteq \mathbb{P}^1(k)$ such that $\varphi_U : U \to V$ is an isomorphism, in particular, $\varphi|_U$ is injective. We observe that for all $x_1 \in k$ such that $x_1 \neq 0$, there are two distinct square roots of x_1^3, i.e., two distinct preimages of $(1 : x_1)$ under φ. Thus if $\varphi|_U$ is injective, we have $U \subseteq V(x_1)$. But $V(x_1) \cap X = \{(0,0)\}$ is not dense in X (because its complement is a nonempty open subset), hence U cannot be dense in X. Thus we reached a contradiction. As a result φ is not birational.

(7) (10 points) Let $\varphi : \mathbb{P}^1(k) \to \mathbb{A}^1(k)$ be a morphism. Prove that there is a point $x \in \mathbb{A}^1(k)$ such that $\varphi(y) = x$ for all $y \in \mathbb{P}^1(k)$.

Solution

Let $\varphi_0 : \mathbb{A}^1(k) \to U_0$ be the isomorphism given by $\varphi_0(u) = (1 : u)$. Then $\varphi|_{U_0} \circ \varphi_0 : \mathbb{A}^1(k) \to \mathbb{A}^1(k)$ is a morphism of affine lines, and hence there is a polynomial $f \in k[u]$ such that $\varphi|_{U_0} \circ \varphi_0(u) = f(u)$ for all $u \in \mathbb{A}^1(k)$. Hence, $\varphi|_{U_0} : U_0 \to \mathbb{A}^1(k)$ is defined by $\varphi|_{U_0}(x_0 : x_1) = f\left(\frac{x_1}{x_0}\right)$. The same argument for the open subset U_1 gives a polynomial $g \in k[u]$ such that $\varphi|_{U_1}(x_0 : x_1) = g\left(\frac{x_0}{x_1}\right)$. Thus $f\left(\frac{x_1}{x_0}\right) = g\left(\frac{x_0}{x_1}\right)$ for all $(x_0 : x_1) \in U_0 \cap U_1$ and in particular $f\left(\frac{x_1}{x_0}\right) = g\left(\frac{x_0}{x_1}\right)$ in $k(\mathbb{P}^1(k))$. We observe that $f\left(\frac{x_1}{x_0}\right) = x_0^{-\deg f}\alpha$ and $g\left(\frac{x_0}{x_1}\right) = x_1^{-\deg g}\beta$ for suitable homogeneous polynomials $\alpha, \beta \in k[x_0, x_1]$ of degrees $\deg f$ and $\deg g$, respectively, such that $x_0 \nmid \alpha$ and $x_1 \nmid \beta$. Since $x_1^{\deg g}\alpha = x_0^{\deg f}\beta$, we conclude that $\deg f = \deg g = 0$, and $f = g$ is an element of k. Thus φ is a constant morphism with value f.