<u>WISB263 Mathematical Statistics</u>
Final Exam
29th June 2021, 15.15-18.15
Total amount of points: 100 + 10 bonus points
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Exercise 1 (25 p.) A number of coins from the reign of King Manuel I Komnenos (1143-1180) were discovered in Cyprus. They arise from four different coinages at intervals throughout his reign. The data below give the silver content ($\%A_g$) of coins. We want to test the hypothesis whether there is any significant difference in their silver content with the passage of time.

The silver content of coins is:

First	Second	Third	**Forth**
5.9	6.91	4.9	5.3
6.8	9.0	5.5	5.6
6.4	6.6	4.6	5.5
7.0	8.1	4.5	5.1
6.61	9.3		6.2
7.7	9.2		5.8
7.2	8.6		5.81
6.9			
6.21			

(i) (4 p.) Is the data from the table (first and forth reign) paired or unpaired? Give arguments to support your answer.

(ii) (8 p.) Perform an appropriate hypothesis test to test whether there is a difference between the silver content of the coins in the **first** and **forth reign** at significance at most $\alpha = 0.05$. Is there evidence that the silver content has been reduced or enhanced?

(iii) (8 p.) Perform the same test as in (ii) assuming that the data is normally distributed with known variance $\sigma^2 = 1$. Compute the p-value. (If you cannot compute the p-value exactly find an upper and lower bound.)

(iv) (5 p.) Compare the findings of the two tests. Which one is more appropriate and has stronger evidence against the null hypothesis? You can assume that the p-value of the first test in (ii) is equal to 0.00034.

Solution

(i) The data is unpaired. The silver content is taken from different coins.

(ii) We perform a Wilcoxon-Mann-Whitney test

First	Forth	Rank	Rank
5.9	5.3	7	2
6.8	5.6	12	4
6.4	5.5	10	3
7.0	5.1	14	1
6.61	6.2	11	8
7.7	5.8	16	5
7.2	5.81	15	6
6.9		13	
6.21		9	

The sum of ranks of the smaller sample is $w = 29$, $\mathbb{E}(W) = 59.5$, we get that the rejection region is equal to

$$\{W < 59.5 + c\} \cup \{W > 59.5 + c\}$$

For $\alpha = 0.05$ we get $40 = 58.5 + c$ so $c = 18.5$ and

$$\{W \leq 40\} \cup \{W \geq 79\}$$

we reject the null hypothesis at $\alpha = 0.05$. There is evidence that the content was reduced.

(iii) We apply Theorem 2.26. We have that $\overline{x}_9 \approx 6.746$ and $\overline{y}_7 \approx 5.616$ and

$$s_{7,9}(\mathbf{x}, \mathbf{y}) \approx \left(\frac{\sqrt{7 \cdot 9}(6.746 - 5.616)}{\sqrt{7+9}} \right)^2 \approx 5.028.$$

We compare the value to $\chi_1^2(0.95) = 3.84$ so we reject the null hypothesis as well. The p-value is equal to 0.0249.

(iv) Both tests agree that there is enough evidence to conclude that the silver content from the first and forth reign is different. Both p-values are small which indicate strong evidence against H_0. The first test is more appropriate since it does not assume that the data is normally distributed, it also has stronger evidence against H_0.

Exercise 2 (25 p.) Let $\mathbf{X} = (X_1, ..., X_n)$ be a random sample such that $X_1 \sim U(\theta, c \cdot \theta)$, where $c > 1$ and $\Theta = \mathbb{R}_+$.

(i) (5 p.) Determine the MLE $\hat{\theta}_n$ of θ.

(ii) (5 p.) Show that the density g_θ of the MLE $\hat{\theta}_n$ is equal to

$$g_\theta(y) = \frac{cn}{(c-1)\theta} \left(\frac{cy - \theta}{(c-1)\theta} \right)^{n-1}$$

for some $y \in D \subset \mathbb{R}$ and 0 otherwise. Determine D.

(iii) (5 p.) Compute $\mathbb{E}_\theta(\hat{\theta}_n)$. Is the estimator unbiased? In case the estimator is biased, determine whether the estimator overestimates or underestimates the true parameter θ.

(iv) (5 p.) Let $c = 3$. We want to design a hypothesis test for testing

$$(H_0) : \theta = 5 \text{ against } (H_1) : \theta \neq 5$$

at significance level $\alpha = 0.05$ using the generalized likelihood ratio statistic $R(\mathbf{X})$. Determine $R(\mathbf{X})$ and the exact rejection region (not the asymptotic one).

(v) (3 p.) Assume that you observe $\{12, 6.1, 8, 5.3, 5, 4.7, 7.6, 11.4, 9, 10.7\}$. Perform the hypothesis test from (iv) in order to decide whether the data was likely coming from a $U(5, 15)$ distribution at significance $\alpha = 0.05$.

(vi) (2 p.) Determine the type-II error when $\theta = 6$ for the data from (v).

Solution:

(i)

$$L(\mathbf{x}; \theta) = \prod_{i=1}^{n} \frac{1}{(c-1)\theta} \mathbb{1}_{\{\theta \leq x_i \leq c\theta\}}$$
$$= \frac{1}{(c-1)^n \theta^n} \mathbb{1}_{\left\{\frac{x_{(n)}}{c} \leq \theta \leq x_{(1)}\right\}}$$

provided that $\frac{x_{(n)}}{c} \leq x_{(1)}$ otherwise 0. The MLE is equal to

$$\hat{\theta}_n = \frac{X_{(n)}}{c}$$

since the likelihood function is decreasing.

(ii) The CDF is equal to

$$\mathbb{P}_\theta(\hat{\theta}_n \leq y) = \left(\frac{cy - \theta}{(c-1)\theta}\right)^n$$

for $y \in [\theta/c, \theta]$ hence $D = (\theta/c, \theta)$. The density follows from differentiating the CDF.

(iii) The expected value is

$$\mathbb{E}_\theta(\hat{\theta}_n) = \frac{1 + cn}{c(n+1)} \theta$$

hence the estimator is biased. The bias is negative hence the estimator underestimates the true parameter.

(iv) We have that

$$R(\mathbf{X}) = \frac{\sup_{\theta \in \Theta} f_\theta(\mathbf{x})}{\sup_{\theta \in \Theta_0} f_\theta(\mathbf{x})} = \frac{10^n}{X_{(n)}^n}.$$

The rejection region $\{R(\mathbf{X}) \geq c\}$ is equivalent to $\{X_{(n)} \leq c'\}$ for some c' satisfying

$$\mathbb{P}_5(X_{(n)} \leq c') = 0.05.$$

Therefore

$$0.05 = \left(\frac{c' - 5}{10}\right)^n - \left(\frac{5 - 5}{10}\right)^n$$

yields that $c' = 5 + 10 \cdot (0.05)^{1/n}$.

(v) From (iv) we know that $c' \approx 12.41$, since $x_{(10)} = 12$ we do reject the null hypothesis.

(vi) The type-II error is equal to

$$1 - \beta_\phi(6) = \mathbb{P}_6(X_{(10)} > 12.41) \approx 0.99.$$

Exercise 3 (25 p.) Consider a random sample $\mathbf{X} = (X_1, ..., X_n)$ with common density

$$f_\theta(x_1) = \frac{\theta}{x_1^{\theta+1}}$$

where $\Theta = (2, \infty)$ and $x_1 > 1$.

(i) (4 p.) Determine the MLE $\hat{\theta}_n$ for θ.

(ii) (2 p.) Is the parametric family of exponential type?

(iii) (5 p.) Design a u.m.p. test for testing $(H_0) : \theta \leq \theta_0$ against $(H_1) : \theta > \theta_0$ at significance α. Describe the rejection region implicitly.

(iv) (2 p.) Show that the Fisher information $I(\theta)$ for the density f_θ is equal to $\frac{1}{\theta^2}$.

(v) (4 p.) Assume that $\hat{\theta}_n$ is asymptotically normal with the corresponding variance (satisfies Theorem 2.11). Determine the Wald random confidence interval for θ for general α.

(vi) (5 p.) Design an asymptotic hypothesis test for

$$(H_0): \ \theta = \theta_0 \text{ against } (H_1) : \theta \neq \theta_0$$

at significance α and determine the rejection region using the information from (v).

(vii) (3 p.) Perform the hypothesis test defined in (vi) when $\theta_0 = 4$ for testing data $\{x_1, ..., x_{100}\}$ such that $\sum_{i=1}^{100} \ln(x_i) = 43.79$ at $\alpha = 0.01$ and determine the confidence interval.

Solution:

(i) $\hat{\theta}_n = \frac{n}{\sum_{i=1}^n \ln(X_i)}$

(ii) Yes, since

$$f_\theta(x_1) = e^{\ln(\theta) - (\theta+1)\ln(x_1)}$$

where $a(\theta) = -(\theta + 1)$ is decreasing.

(iii) By Theorem 2.19 we have that

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n \ln(X_i) < c \\ 0 & \text{else} \end{cases}$$

and

$$\alpha = \mathbb{P}_{\theta_0}\left(\sum_{i=1}^n \ln(X_i) < c\right).$$

We reject if $\sum_{i=1}^n \ln(x_i) < c$.

(iv)

$$I(\theta) = \frac{1}{\theta^2}$$

since $\frac{\partial^2}{\partial\theta^2}\ln(f_\theta(x_1)) = -\frac{1}{\theta^2}$.

(v) The Wald random estimator is equal to

$$I(\hat{\theta}_n) = \left(\frac{\sum_{i=1}^n \ln(X_i)}{n}\right)^2$$

and the asymptotic random confidence interval is equal to

$$\left[\frac{n}{\sum_{i=1}^n \ln(X_i)} - z(\alpha/2)\frac{\sqrt{n}}{\sum_{i=1}^n \ln(X_i)}, \frac{n}{\sum_{i=1}^n \ln(X_i)} + z(\alpha/2)\frac{\sqrt{n}}{\sum_{i=1}^n \ln(X_i)}\right].$$

(vi) By the duality relation between confidence intervals and acceptence regions in hypothesis testing we obtain that we can design a test

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \hat{\theta}_n < c_1 \text{ or } \hat{\theta}_n > c_2 \\ 0 & \text{otherwise}, \end{cases}$$

where $c_1 = \frac{4}{1+\frac{z(\alpha/2)}{\sqrt{n}}}$ and $c_2 = \frac{4}{1-\frac{z(\alpha/2)}{\sqrt{n}}}$.

(vii) We have that $z(0.005) = 2.57$, $c_1 = 3.18$ and $c_2 = 5.38$. The MLE is equal to $\hat{\theta}_{100} = 2.28$. We reject the null hypothesis. The confidence interval is equal to $[1.69, 2.87]$.

Exercise 4 (25 p.) Consider independent random variables $Z_1, ..., Z_n$ such that

$$Z_i \sim N(\mu, \sigma^2(1 + x_i))$$

and $x_i > -1$ for all $i = 1, ..., n$.

(i) (5 p.) Write the random variables $Y_1, ..., Y_n$ in terms of $Z_1, ..., Z_n$ such that for all $i = 1, ..., n$

$$Y_i = a_i + \epsilon_i$$

where $\epsilon_i \sim N(0, \sigma^2)$. Determine a_i for all $i = 1, ..., n$.

(ii) (7 p.) What is the distribution of $\mathbf{Y} = (Y_1, ..., Y_n)$? Compute directly the unbiased MLE's $\hat{\mu}_n$ and $S^2_{\overline{Y}_n}$ for μ resp. σ^2 for the sample $\mathbf{Y}$.

(iii) (5 p.) Assume first that $\frac{n-1}{\sigma^2}S^2_{\overline{Y}_n}$ is χ^2_{n-1} distributed. Discuss the hypothesis test

$$(H_0): \sigma = 2 \text{ against } (H_1): \sigma < 2.$$

Which test statistic could you choose? Determine the rejection region for $\alpha = 0.05$ and $n = 10$. For which $s^2_{\overline{y}_{10}}$ do we accept H_0?

(iv) (3 p.) Find a matrix $\mathbf{X}$ and B such that we can write

$$\mathbf{Y} = \mathbf{X}B + \epsilon_n$$

where $\epsilon_n \sim N_n(0, \sigma^2 Id)$. How do you call $\mathbf{Y}$ written in this way?

(v) (5 p.) Show that $S^2_{\bar{Y}_n}$ is χ^2_{n-1} distributed.

Solution:

(i) We can write first of all

$$Z_i = \mu + \epsilon_i'$$

and $\epsilon_i' \sim N(0, \sigma^2(1+x_i))$ and then the linear model as

$$Y_i = \frac{\mu}{\sqrt{1+x_i}} + \epsilon_i$$

setting $Y_i = \frac{Z_i}{\sqrt{1+x_i}}$ and where $\epsilon_i \sim N(0, \sigma^2)$.

(ii) $Y_i \sim N(\frac{\mu}{\sqrt{1+x_i}}, \sigma^2)$. We get that

$$\hat{\mu}_n = \frac{\sum_{i=1}^n Y_i/\sqrt{1+x_i}}{\sum_{i=1}^n 1/(1+x_i)}$$

and $S^2_{\bar{Y}_n} = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \hat{\mu}_n)^2$.

(iii) We can take as test statistic $\frac{(n-1)}{4} S^2_{\bar{Y}_n}$. The test rejects for small values of σ. The rejection region is equal to

$$\left\{ S^2_{\bar{Y}_{10}} \leq \frac{4}{9}\chi^2_{10-1}(0.95) \right\} = \{S^2_{\bar{Y}_{10}} \leq 7.52\}.$$

We accept for all $s^2_{\bar{y}_{10}} > 7.52$.

(iv) The design matrix is equal to the vector $(1/(1+x_1), ..., 1/(1+x_n))^T$ and $B = \mu$. We have a simple regression model.

(v) Follows from Theorem 2.17 that $\frac{n-1}{\sigma^2} S^2_{\bar{Y}_n} \sim \chi^2_{n-1}$.

BONUS (10 p.) Consider a two-sided test with critical region $\{S_n(\mathbf{X}) \leq c_1\} \cup \{S_n(\mathbf{X}) \geq c_2\}$. Prove that the p-value for this test is uniformly distributed over $[0,1]$.

Solution: The p-value for a two-sided test is equal to

$$p(s(\mathbf{x})) = 2\min\{F_S(s(\mathbf{x})), 1 - F_S(s(\mathbf{x}))\}.$$

Let $y \in (0,1)$, then

$$\begin{aligned}
\mathbb{P}(p(S(\mathbf{X})) \leq y) &= \mathbb{P}(\min\{F_S(S(\mathbf{X})), 1 - F_S(S(\mathbf{X}))\} \leq y/2) \\
&= \mathbb{P}(F_S(S(\mathbf{X})) \leq y/2, F_S(S(\mathbf{X})) \leq 1/2) + \mathbb{P}(F_S(S(\mathbf{X})) \geq 1 - y/2, F_S(S(\mathbf{X})) > 1/2) \\
&= \frac{y}{2} + \frac{y}{2} = y.
\end{aligned}$$