Solution 1

(a) By the chain rule for differentiation it follows that g_t is C^1 with derivative given by

$$\frac{d}{dr}g_t(r) = Df(r\sigma(t))\sigma(t) = \langle \mathrm{grad} f(r\sigma(t))\,,\,\sigma(t)\rangle.$$

It follows from the given estimate that

$$r\frac{d}{dr}g_t(r) \geq \|r\sigma(t)\|^2 = r^2, \qquad (r \geq 0).$$

This implies dat $(g_t)'(r) \geq r$ for $r \geq 0$, hence it follows that g_t is strictly increasing on $[0,\infty[$. Furthermore, $g_t(r) - \frac{1}{2}r^2$ has nonnegative derivative, hence is monotonically increasing on $[0,\infty[$. It follows that $g_t(r) \geq g_t(0) + \frac{1}{2}r^2 = -1 + \frac{1}{2}r^2$, for $r \geq 0$.

(b) From (a) it follows that $g_t(r) > 0$ for $r > \sqrt{2}$. By continuity, g_t attains the value 0 on $]0,\infty[$, and since g_t is strictly increasing, this happens in a unique $\rho(t) > 0$. Now $g_t(r) = 0$ is equivalent to $r\sigma(t) \in M$. The result follows.

(c) We consider the function $F : (r,t) \mapsto f(r\sigma(t))$ on the open subset $\Omega =]0,\infty[\times\mathbb{R}$ of $\mathbb{R}^2$. Fix $t_0 \in \mathbb{R}$ and put $r_0 = \rho(t_0))$. Then $(r_0,t_0) \in \Omega$ and

$$D_1F(r_0,t_0) = \frac{d}{dr}g_t(r_0) > 0,$$

so by the implicit function theorem there exist an open neighborhood U of t_0 in $\mathbb{R}$ and an open neighborhood V of r_0 in $]0,\infty[$ such that

- for each $t \in U$ there exists a unique $r = r(t) \in V$ such that $F(r(t),t) = 0$;
- the function $t \mapsto r(t)$ is C^1 on U.

By the uniqueness statement of (b) it follows that $r(t) = \rho(t)$ for $t \in U$, which shows that ρ is C^1 in an open neighborhood of t_0. Since t_0 was arbitrary, the result follows.

(d) By differentiation of $f(\rho(t)\sigma(t)) = 0$ with respect to t it follows by application of the chain rule that

$$\langle \mathrm{grad} f(\rho(t)\sigma(t))\,,\,\rho'(t)\sigma(t) + \rho(t)\sigma'(t)\rangle = 0.$$

This implies the required identity.

Solution 2

(a) This follows by one of the characterizations of submanifold, see book [DK1, Thm 4.7.1 (iii)].

(b) The map i is continuous $\mathbb{R}^{n-1} \to \mathbb{R}^n$ and U is open in $\mathbb{R}^n$. Therefore, $i^{-1}(U)$ is open in $\mathbb{R}^{n-1}$.

(c) Since i and f are C^1, it follows by application of the chain rule that the composition $f \circ i$ is C^1 as well. Furthermore,

$$D(f \circ i)(0) = Df(i(0)) \circ Di(0) = Df(0) \circ i.$$

The latter identity holds since i is linear. Since f is submersive at 0, we have $Df(0)(\mathbb{R}^n) = \mathbb{R}^{n-p}$. By hypothesis,

$$i(\mathbb{R}^{n-1}) + T_0M = \mathbb{R}^n.$$

By the various characterizations of tangent space, $T_0M = \ker Df(0)$, and we see that $Df(0)i(\mathbb{R}^{n-1}) = Df(0)(\mathbb{R}^n)$. Therefore,

$$D(f \circ i)(0)(\mathbb{R}^{n-1}) = Df(0)i(\mathbb{R}^{n-1}) = Df(0)(\mathbb{R}^n) = \mathbb{R}^{n-p}.$$

We conclude that $f \circ i$ is submersive at i.

(d) From (c) it follows that $(f \circ i)^{-1}(0)$ is a submanifold of dimension $n-1-(n-p) = p-1$ at the point 0 in $\mathbb{R}^{n-1}$. Hence, there exists an open neighborhood U_1 of 0 in $U_0 \subset \mathbb{R}^{n-1}$ such that $U_1 \cap (f \circ i)^{-1}(0)$ is a submanifold of $\mathbb{R}^{n-1}$ of dimension $p-1$. Now

$$\begin{aligned} U_1 \cap (f \circ i)^{-1}(0) &= \{x \in U_1 \mid f(i(x)) = 0\} \\ &= \{x \in U_1 \mid f(x,0) = 0\} \\ &= \{x \in U_1 \mid (x,0) \in M\} = N. \end{aligned}$$

Solution 3

(a) The functions $g : x \mapsto 0$ and $f : x \mapsto h$ are continuous on A. Since $A \in \mathscr{J}(\mathbb{R}^n)$ it follows that

$$A \times [0,h] = \{(x,y) \mid x \in A, g(x) \leq y \leq f(x)\}$$

is compact Jordan measurable. Furthermore,

$$\mathrm{vol}_{n+1}(A \times [0,h]) = \int_A [f(x) - g(x)]\, dx = h \int_A dx = h\mathrm{vol}_n(A).$$

(b) The map Φ is partially differentiable with continuous partial derivatives, hence C^1. It obviously maps U to itself. The map $\Psi : U \to \mathbb{R}^{n+1}$ defined by $\Psi(y,s) = ((1-s)^{-1}y, s)$ maps U into itself and is readily checked to be C^1 and a two-sided inverse for Φ. It follows that Φ is a C^1 diffeomorphism from U onto itself. Its total derivative has $(n+1)\times(n+1)$ matrix

$$D\Phi(x,t) = \begin{pmatrix} (1-t)I_n & -x \\ 0 & 1 \end{pmatrix}.$$

The associated Jacobian equals

$$\det D\Phi(x,t) = \det((1-t)I_n) = (1-t)^n.$$

(c) It follows from the definition of C_A that

$$C_A(\delta) = \Phi(A \times [0, 1-\delta]).$$

This implies that $C_A(\delta)$ is compact Jordan measurable in $\mathbb{R}^{n+1}$. Furthermore, by substitution of variables we have

$$\begin{aligned}
\mathrm{vol}_{n+1}(C_A(\delta)) &= \int_{\Phi(A(\delta))} d(y,s) = \int_{A(\delta)} |\det D\Phi(x,t)|\, d(x,t) \\
&= \int_{A\times[0,1-\delta]} (1-t)^n\, d(x,t) = \int_{\mathbb{R}^{n+1}} 1_A(x) 1_{[0,1-\delta]}(t)(1-t)^n\, d(x,t) \\
&= \int_{\mathbb{R}^n} \overline{\int_{\mathbb{R}}} 1_A(x) 1_{[0,1-\delta]}(t)(1-t)^n\, dt dx = \mathrm{vol}_n(A) \int_0^{1-\delta} (1-t)^n\, dt \\
&= (n+1)^{-1}(1-\delta^{n+1})\mathrm{vol}_n(A).
\end{aligned}$$

(d) We note that

$$C_A(\delta) \subset C_A \subset C_A(\delta) \cup (A \times [1-\delta, 1])$$

where the sets on the left and on the right are compact Jordan measurable. It follows that

$$1_{C_A(\delta)} \leq 1_{C_A} \leq 1_{C_A(\delta)\cup(A\times[1-\delta,1])}.$$

Hence:

$$\mathrm{vol}(C_A(\delta)) = \underline{\int} 1_{C_A(\delta)} \leq \underline{\int} 1_{C_A} \leq \overline{\int} 1_{C_A} \leq \overline{\int} 1_{C_A(\delta)\cup(A\times[1-\delta,1])} = \mathrm{vol}(C_A(\delta)) + \delta \mathrm{vol}_n(A).$$

Taking the limit for $\delta \downarrow 0$ we infer that 1_{C_A} is Riemann integrable over $\mathbb{R}^{n+1}$ with integral equal to

$$\lim_{\delta\downarrow 0} \mathrm{vol}_{n+1}(C_A(\delta)) = (1+n)^{-1}\mathrm{vol}_n(A).$$

Solution 4

(a) For each point $a \in \operatorname{supp}\chi$ there exists an open neighborhood U_a of a in $\mathbb{R}^n$ such that $U_a \cap M$ is a coordinate chart. Let $\mathscr{U}$ be the collection of the sets U_a for $a \in \operatorname{supp}\chi$. Then by compactness of $\operatorname{supp}\chi$ it follows that there exists a continuous partition of unity $\{\psi_j \mid 1 \leq j \leq N\}$ over $\operatorname{supp}\chi$ which is subordinate to $\mathscr{U}$. By assumption the lemma is valid with $\psi_j|_M \cdot \chi$ in place of χ, for every j. Put

$$F_j(t) = \int_M \psi_j(x)\chi(x) f(t,x)\, d_k x.$$

Then F_j is differentiable with derivative

$$F_j'(t) = \int_M \psi_j(x)\chi(x) D_1 f(t,x)\, d_k x.$$

Now $F = \sum_{j=1}^N F_j$, hence F is differentiable with derivative

$$F'(t) = \sum_{j=1}^N F_j'(t) = \int_M \sum_j \psi_j(x)\chi(x) D_1 f(t,x)\, dx = \int_M \chi(x) D_1 f(t,x)\, dx.$$

(b) We may use that $B(0;1)$ is an open subset of $\mathbb{R}^n$ with C^1-boundary. As the boundary is closed and bounded, it is compact. Hence the constant function $\chi = 1$ belongs to $C_c(\partial B(0;1))$.

The function $(t,x) \mapsto h(tx)$ is continuous and partially differentiable with respect to t, with derivative

$$\frac{d}{dt} h(tx) = \langle \operatorname{grad} h(tx)\,,\, x\rangle.$$

The latter is continuous in $(t,x) \in \mathbb{R} \times \partial B(0;1)$. Applying the lemma with $\chi = 1$ we see that the function

$$I(t) := \int_{\partial B(0;1)} h(tx)\, d_{n-1}x$$

is differentiable in $t \in \mathbb{R}$ with derivative:

$$I'(t) = \int_{\partial B(0;1)} \langle \operatorname{grad} h(tx)\,,\, x\rangle\, d_{n-1}x.$$

Now $B(0;1)$ is an open set with C^1-boundary, and the outward unit normal on the boundary is given by $\mathbf{n}(x) = x$ for $x \in \partial B(0;1)$. This implies the asserted identity.

(c) We now observe that $v_t : x \mapsto \operatorname{grad} h(tx)$ is a C^1-vector field on $\mathbb{R}^n$ with divergence $\operatorname{div}(v_t) = t \operatorname{div} \operatorname{grad} h(tx)$. This shows that

$$\operatorname{div}(v_t)(x) = 0, \qquad (x \in B(0;1),\ |t| \leq 1).$$

It now follows by application of the divergence theorem that

$$\int_{\partial B(0;1)} \langle \operatorname{grad} h(tx)\,,\, x\rangle \, d_{n-1}x = \int_{B(0;1)} \operatorname{div}(v_t)(x)\, dx = 0$$

for $t \in [-1,1]$. It follows that the function I is constant on $[-1,1]$ hence $I(1) = I(0)$ and we obtain

$$\int_{\partial B(0;1)} h(x)\, d_{n-1}x = I(0) = h(0) \int_{\partial B(0;1)} d_{n-1}x = h(0)\mathrm{vol}_{n-1}(\partial B(0;1)).$$

From this the required result follows.