Exam Analyse in Meer Variabelen 2021-06-28, 11:30 - 14:30

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- This is an open book exam: you may use the books and the extra notes and your personal notes.
- Justify your anwers with complete arguments, unless specified otherwise. If you use results from the books or lecture notes, always **refer to them**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score through division by 4, but not higher than 10.
- You are free to write the solutions either in English, or in Dutch.

Good Luck !

10 pt total **Exercise 1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that for all $x \in \mathbb{R}^2$ we have $\langle \mathrm{grad} f(x) \, , x \rangle \geq \|x\|^2$. We assume that $f(0) = -1$ and put

$$M := \{x \in \mathbb{R}^2 \mid f(x) = 0\}.$$

Let $\sigma : \mathbb{R} \to \mathbb{R}^2$ be given by $\sigma(t) := (\cos t, \sin t)$.

3 pt (a) Show that for every $t \in \mathbb{R}$ the function $g_t : r \mapsto f(r\sigma(t))$ is strictly increasing on $[0, \infty[$, with $g_t(r) \geq -1 + \frac{1}{2}r^2$ for all $r \geq 0$.

1 pt (b) Show that for every $t \in \mathbb{R}$ there exists a unique $\rho(t) > 0$ such that $\rho(t)\sigma(t) \in M$.

4 pt (c) By using the implicit function theorem, show that $t \mapsto \rho(t)$ is a C^1 function on $\mathbb{R}$ with values in $]0, \infty[$.

2 pt (d) Show that
$$\rho'(t) = -\rho(t) \frac{\langle \mathrm{grad} f(\rho(t)\sigma(t)) \, , \sigma'(t) \rangle}{\langle \mathrm{grad} f(\rho(t)\sigma(t)) \, , \sigma(t) \rangle}, \quad (t \in \mathbb{R}).$$

10 pt total **Exercise 2.** Let M be a C^1-submanifold of dimension $p \geq 1$ in $\mathbb{R}^n$ containing 0. Let $i : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the inclusion map $x \mapsto (x, 0)$. We assume that $i(\mathbb{R}^{n-1}) + T_0M = \mathbb{R}^n$.

1 pt (a) Argue that there exists an open neighborhood U of 0 in $\mathbb{R}^n$ and a C^1 submersion $f : U \to \mathbb{R}^{n-p}$ such that $U \cap M = f^{-1}(0)$.

1 pt (b) Show that the pre-image $U_0 := i^{-1}(U)$ is open in $\mathbb{R}^{n-1}$ and contains 0.

4 pt (c) Show that $f \circ i : U_0 \to \mathbb{R}^{n-p}$ is C^1 and submersive at 0.

4 pt (d) Show that there exists an open neighborhood $U_1 \subset U_0$ of 0 in $\mathbb{R}^{n-1}$ such that $N := \{x \in U_1 \mid (x, 0) \in M\}$ is a submanifold of $\mathbb{R}^{n-1}$ of dimension $p - 1$.

12 pt total **Exercise 3.** We assume that A is a compact Jordan measurable subset of $\mathbb{R}^n$.

2 pt (a) Show that for every $h \geq 0$ the set $A \times [0,h]$ is compact Jordan measurable in $\mathbb{R}^{n+1}$ with volume given by $\mathrm{vol}_{n+1}(A \times [0,h]) = h\,\mathrm{vol}_n(A)$.

3 pt (b) Show that the map $\Phi : (x,t) \mapsto ((1-t)x, t)$ is a C^1 diffeomorphism from $U := \mathbb{R}^n \times]-1,1[$ onto itself, and compute the associated Jacobian determinant.

We define C_A to be the cone in $\mathbb{R}^{n+1}$ consisting of the points $(y,s) \in \mathbb{R}^n \times [0,1]$ such that $y \in (1-s)A$. Furthermore, for $0 < \delta < 1$ we put

$$C_A(\delta) := \{(y,s) \in C_A \mid 0 \leq s \leq 1-\delta\}.$$

4 pt (c) Show that $\mathrm{vol}_{n+1}(C_A(\delta)) = (1-\delta^{n+1})(n+1)^{-1}\mathrm{vol}_n(A)$.

3 pt (d) Show that C_A is compact Jordan measurable in $\mathbb{R}^{n+1}$ with volume given by

$$\mathrm{vol}_{n+1}(C_A) = (n+1)^{-1}\mathrm{vol}_n(A).$$

12 pt total **Exercise 4.** The following lemma is central in this exercise.

Lemma *Let $I \subset \mathbb{R}$ be an open interval and M a k-dimensional C^1-submanifold of $\mathbb{R}^n$. Let $f : I \times M \to \mathbb{R}$ be a continuous function on $I \times M$ which is partially differentiable with respect to the first variable, with associated continuous partial derivative $D_1 f : I \times M \to \mathbb{R}$. Let $\chi \in C_c(M)$. Then the function $F : I \to \mathbb{R}$ defined by*

$$F(t) = \int_M \chi(x) f(t,x)\, d_k x$$

is differentiable with derivative

$$F'(t) = \int_M \chi(x) D_1 f(t,x)\, d_k x.$$

If M is an open subset of Euclidean space, you may have seen a proof in a previous course. As a consequence, you may use that for general M, the lemma holds whenever χ is supported in a chart of M.

4 pt (a) Prove the lemma.

In the following you may use without proof that the open unit ball $B(0;1)$ is an open domain with C^1-boundary in $\mathbb{R}^n$.

4 pt (b) If $h : \mathbb{R}^n \to \mathbb{R}$ is a C^2-function, show that

$$\frac{d}{dt}\int_{\partial B(0;1)} h(tx)\, d_{n-1}x = \int_{\partial B(0;1)} \langle \mathrm{grad}\, h(tx)\,,\, \mathbf{n}(x)\rangle\, d_{n-1}x, \quad (t \in \mathbb{R}).$$

4 pt (c) If in addition $\Delta h = \mathrm{div}\,\mathrm{grad}\, h = 0$ on $B(0;1)$, show that

$$h(0) = \mathrm{vol}_{n-1}(\partial B(0;1))^{-1} \int_{\partial B(0;1)} h(x)\, d_{n-1}x.$$