Solution 1

(a) By the result mentioned in the beginning, F is continuous. Fix $x \in \mathbb{R}$. Then by the fundamental theorem of integration it follows that $F(x,\cdot)$ is differentiable with derivative given by

$$D_2F(x,y) = y^{-1}e^{xy}, \qquad (y > 0).$$

By differentiation under the integral sign, it follows that

$$D_1F(x,y)) = \int_1^y e^{xt}\, dt = \frac{e^{xy} - e^x}{x}.$$

It readily follows from these expressions that D_jF are continuous on $J \times J$, for $j = 1,2$. Hence, F is C^1.

(b) From $t^{-1}e^{xt} > 0$ it follows that $F(x,y)$ is strictly increasing in y. From $e^{xt}/t \to \infty$ for $t \to \infty$, we see that there exists $y_1 > 1$ such that $F(x,y_1) > 1$. Since $F(x,1) = 0$ it follows by the intermediate value theorem applied to the continuous function $F(x,\cdot)$ that there exists an element $\eta(x) \in]1,y_1[$ such that $F(x,\eta(x)) = 1$. Since $y \mapsto F(x,y)$ is strictly increasing, the element $\eta(x)$ is uniquely determined.

(c) Let $x_0 \in J$ be fixed and put $y_0 = \eta(x_0)$. Then $D_2F(x_0,y_0) = y_0^{-1}e^{x_0y_0} > 0$. Since F is C^1 it follows by the implicit function theorem applied to $F - 1$ that there exist open neighborhoods U of x_0 in J and V of y_0 in J such that for each $x \in U$ there exists a unique $y(x) \in V$ such that $F(x,y(x)) - 1 = 0$ and such that the function $x \mapsto y(x)$ is C^1 on U. By the uniqueness of (b) it follows that $y(x) = \eta(x)$ for $x \in U$. Hence, η is C^1 on U. Since $x_0 \in J$ was arbitrary, it follows that $\eta : J \to \mathbb{R}$ is C^1.

(d) It follows by application of the implicit function theorem that

$$\begin{aligned}
\eta'(x) &= -D_2F(x,\eta(x))^{-1}D_1F(x,\eta(x)) \\
&= -\eta(x)e^{-x\eta(x)}\frac{e^{x\eta(x)} - e^x}{x} \\
&= \frac{\eta(x)}{x}(e^{x(1-\eta(x))} - 1).
\end{aligned}$$

Solution 2

(a) Since M is a p-dimensional submanifold, there exists an open neighorhood $U \ni a$ in $\mathbb{R}^n$ and a submersion $g : U \to \mathbb{R}^{n-p}$ such that $M \cap U = g^{-1}(0)$. It is a theorem that in this case

$$T_aM = \ker Dg(a).$$

(b) We must show that $Dh(a) : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear map. For this it suffices to show that $Dh(a)$ has a trivial kernel. Let $\xi \in \mathbb{R}^n$ and assume $Dh(a)\xi = 0$. Since

$$Dh(a) = \begin{pmatrix} Dg(a) \\ Df(a) \end{pmatrix}$$

as $n \times n$ matrices, it follows that $Df(a)\xi = 0$ and $Dg(a)\xi = 0$. From the second equation it follows that $\xi \in T_aM$. Since $Df(a)|_{T_aM}$ is injective, it now follows that $\xi = 0$. Then $Dg(a)\xi = 0$ and $Df(a)\xi = 0$ hence $\xi \in T_aM$ and $Df(a)\xi = 0$. Since T_aM is one dimensional, and $Df(a)$ is not identically zero on T_aM, it follows that $\xi = 0$.

(c) It follows from the local inverse function theorem that there exists an open neighborhood U_0 of a in U such that h maps U_0 diffeomorphically onto an open subset V of $\mathbb{R}^n$. We have that $h(a) = 0$ so $0 \in V$ and by injectivity of h on U_0 it follows that $h^{-1}(\{0\}) \cap U_0 = \{a\}$.

(d) By definition of N we have

$$N \cap U_0 = f^{-1}(\{0\}) \cap M \cap U_0.$$

By (a) we have $M \cap U_0 = g^{-1}(\{0\}) \cap U_0$. It follows that

$$N \cap U_0 = h^{-1}(\{0\}) \cap U_0.$$

Now use (c).

Solution 3

(a) By a simple calculation we find that

$$\psi(s,t) = R_{\sigma(t)}\varphi(s) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix} \begin{pmatrix} \varphi_1(s) \\ \varphi_2(s) \\ 0 \end{pmatrix}.$$

It follows that

$$\psi(s,t) = \begin{pmatrix} \varphi_1(s)\cos t \\ \varphi_2(s) \\ -\varphi_1(s)\sin t \end{pmatrix},$$

hence

$$D_1\psi(s,t) = \begin{pmatrix} \varphi_1'(s)\cos t \\ \varphi_2'(s) \\ -\varphi_1'(s)\sin t \end{pmatrix}, \quad \text{and} \quad D_2\psi(s,t) = \begin{pmatrix} -\varphi_1(s)\sin t \\ 0 \\ -\varphi_1(s)\cos t \end{pmatrix}.$$

These partial derivatives are continuous functions, hence ψ is C^1. It is readily verified that $D_1\psi(s,t) \perp D_2\psi(s,t)$, so that

$$\begin{aligned}\|D_1\psi(s,t)\times D_2\psi(s,t)\| &= \|D_1\psi(s,t)\|\cdot\|D_2\psi(s,t)\| \\ &= |\varphi_1(s)|\|\varphi'(s)\| = \varphi_1(s)\|\varphi'(s)\|.\end{aligned}$$

The latter follows since $\varphi(s)\in M$ so that $\varphi(s)_1 > 0$. Of course, the result can also be obtained by calculating all components of the exterior product.

Since $\varphi_1(s)>0$ and $\varphi'(s)\neq 0$ it follows from the above that $D_1\psi(s,t)\times D_2\psi(s,t)\neq 0$. This implies that for al that $(s,t)\in]a,b[\times]\alpha,\beta[$ the linear map $D\psi(s,t)$ has rank 2 hence is injective. Hence ψ is an immersion.

(b) Since every point of N has an open neighborhood which is the image of an embedding ψ as above it follows that N is a 2-dimensonal submanifold of $\mathbb{R}^3$. Since N is the image of the compact subset $S\times M$ of $\mathbb{R}^4$ by the continuous map $(u,x)\mapsto R_u(x,0)^{\mathrm{T}}$ it follows that N is compact.

(c) By definition,

$$\int_N f(y)\, d_2y = \int_{]a,b[\times]\alpha,\beta[} f(\psi(s,t))\|D_1\psi(s,t)\times D_2\psi(s,t)\| ds dt.$$

By applying (a) we find

$$\begin{aligned}\int_N f(y)\, d_2y &= \int_{]a,b[\times]\alpha,\beta[} f(R_{\sigma(t)}(\varphi(s),0)^{\mathrm{T}})\varphi_1(s)\|\varphi'(s)\| ds dt \\ &= \int_\alpha^\beta\left(\int_a^b (\varphi(s))_1 f(R_{\sigma(t)}[(\varphi(s),0)^{\mathrm{T}}])\|\varphi'(s)\|\, ds\right)\, dt \\ &= \int_\alpha^\beta\int_M x_1 f(R_{\sigma(t)}[(x,0)^{\mathrm{T}})]\, d_1x\, dt \\ &= \int_S\int_M x_1 f([R_u(x,0)^{\mathrm{T}}])\, d_1x\, d_1u.\end{aligned}$$

In the last equality we have used that $\|\sigma'(t)\| = 1$.

(d) The set N is compact and covered by open subsets $U\subset\mathbb{R}^3$ such that $U\cap N$ is the image of an embedding. Let $(\psi_j)_{1\leq j\leq N}$ be a continuous partition of unity over N subordinate to this covering. Then (c) holds with $f=\psi_j|_N$ for every j. By additivity of integration it follows that (c) is valid with $f=1$. Therefore,

$$\mathrm{vol}_2(N) = \int_N 1\cdot d_2y = \int_S\int_M x_1\cdot 1\cdot d_1x\, d_1u = \mathrm{vol}_1(S)\int_M x_1\, d_1x$$

and the result follows.

Solution 4

(a) For $x \in \mathbb{R}^3 \setminus \{0\}$ we have $\frac{\partial}{\partial x_j}\|x\| = \|x\|^{-1}x_j$. Hence,

$$D_j v_j(x) = -\frac{\partial}{\partial x_j}\frac{x_j}{\|x\|^3} = -\|x\|^{-3} + 3x_j\|x\|^{-5}x_j.$$

Summing this over $1 \leq j \leq 3$ we find

$$\operatorname{div} v(x) = -3\|x\|^{-3} + 3\|x\|^2\|x\|^{-5} = 0.$$

(b) We note that on $\mathbb{R}^3 \setminus \{0\}$ we have $D_j((fv)_j) = D_j(fv_j) = D_j f v_j + f D_j v_j$. Summing over $1 \leq j \leq 3$ we find

$$\operatorname{div}(fv) = \langle \operatorname{grad} f\,,\, v\rangle + f \operatorname{div} v.$$

Now use (a) to conclude that the last term vanishes.

(c) We note that $\Omega_\delta := B(0;R) \setminus \bar{B}(0;\delta)$ is a bounded domain with C^1-boundary. Furthermore, fv is a C^1-function defined on an open neighborhood of Ω_δ. By application of the divergence theorem it follows that

$$I_\delta = \int_{\partial\Omega_\delta} \langle f(y)v(y)\,,\, \mathbf{n}(y)\rangle\, d_2y,$$

where $\mathbf{n}$ denotes the outward unit normal on $\partial\Omega_\delta$. The latter boundary is equal to the disjoint union of the boundaries $\partial B(0;\delta)$ and $\partial B(0;R)$. On the latter part the function f vanishes. For $y \in \partial B(0;\delta)$ we have $\mathbf{n}(y) = -y/\delta$. We conclude that

$$\begin{aligned} I_\delta &= \int_{\partial B(0;\delta)} \langle f(y)v(y)\,,\, \mathbf{n}(y)\rangle\, d_2y \\ &= \int_{\partial B(0;\delta)} \langle f(y)v(y)\,,\, -y/\delta\rangle\, d_2y \\ &= \int_{\partial B(0;\delta)} f(y)\frac{1}{\delta\|y\|}\, d_2y \\ &= \frac{1}{\delta^2}\int_{\partial B(0;\delta)} f(y) d_2y. \end{aligned}$$

(d) Let $\varepsilon > 0$. By continuity we may fix $\delta_0 > 0$ such that $|f - f(0)| < \varepsilon/5\pi$ on $B(0;\delta_0)$. Then for $0 < \delta < \delta_0$ we have

$$\left|\int_{\partial B(0;\delta)} (f(y) - f(0))\, d_2y\right| \leq \operatorname{vol}_2(B(0;\delta))\varepsilon/5\pi < \delta^2\varepsilon.$$

It follows from this that $\delta^2|I_\delta - 4\pi f(0)| < \delta^2\varepsilon$. Thus, for all $0 < \delta < \delta_0$ we have

$$|I_\delta - 4\pi f(0)| < \varepsilon.$$

The result follows.