Retake Analyse in Meer Variabelen 2021-07-19, 11:30 - 14:30

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- This is an open book exam: you may use the books and the extra notes and your personal notes.
- Justify your anwers with complete arguments, unless specified otherwise. If you use results from the books or lecture notes, always **refer to them**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score through division by 4, but not higher than 10.
- You are free to write the solutions either in English, or in Dutch.

Good Luck !

10 pt total **Exercise 1.** Put $J =]0,\infty[$. In this exercise you may use that if $g : J \times J \to \mathbb{R}$ is a continuous function, then $(x,y) \mapsto \int_1^y g(x,t)\,dt$ is a continuous function $J \times J \to \mathbb{R}$ as well.

2 pt (a) Show that the function $F : J \times J \to \mathbb{R}$ defined by

$$F(x,y) = \int_1^y t^{-1} e^{xt}\,dt$$

is C^1. Determine the partial derivatives D_1F and D_2F.

2 pt (b) Show that for every $x > 0$ there exists a unique $\eta(x) > 1$ such that $F(x,\eta(x)) = 1$. Hint: use that $e^{xt}/t \to \infty$ for $t \to \infty$.

4 pt (c) Show that the function $\eta : J \to \mathbb{R}$ is C^1.

2 pt (d) Show that for all $x > 0$ we have

$$\eta'(x) = \frac{\eta(x)}{x}(e^{x(1-\eta(x))} - 1).$$

10 pt total **Exercise 2.** Let $M \subset \mathbb{R}^n$ be a p-dimensional submanifold $(1 \le p < n)$. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be C^1-function and put $N = \{x \in M \mid f(x) = 0\}$. We fix $a \in N$ and assume that the restriction $Df(a)|_{T_aM} : T_aM \to \mathbb{R}^p$ is injective.

2 pt (a) Show that there exists an open neighborhood $U \ni a$ in $\mathbb{R}^n$ and a C^1-map $g : U \to \mathbb{R}^{n-p}$ such that $M \cap U = g^{-1}(\{0\})$ and $T_aM = \ker Dg(a)$.

Define the map $h : U \to \mathbb{R}^n$ by $h := \begin{pmatrix} f \\ g \end{pmatrix}$.

3 pt (b) Show that $Dh(a) \in \mathrm{Aut}(\mathbb{R}^n)$. Hint: consider the kernel of $Dh(a)$.

2 pt (c) Show that there exists an open neighborhood $U_0 \ni a$ contained in U such that $h^{-1}(\{0\}) \cap U_0 = \{a\}$.

3 pt (d) Show that $N \cap U_0 = \{a\}$.

12 pt total **Exercise 3.** Let M be a 1-dimensional submanifold of $\mathbb{R}^2$, contained in the halfplane $\{x \in \mathbb{R}^2 \mid x_1 > 0\}$. Let $S = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$, the unit circle in $\mathbb{R}^2$. For $u = (u_1,u_2) \in S$ we define

$$R_u = \begin{pmatrix} u_1 & 0 & u_2 \\ 0 & 1 & 0 \\ -u_2 & 0 & u_1 \end{pmatrix}.$$

Substituting $\sigma(t) := (\cos t, \sin t)$, with $t \in \mathbb{R}$, for u, we see that $R_{\sigma(t)} : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation about the x_2-axis by angle t. We define

$$N := \{R_u[(x,0)^{\mathrm{T}}] \ \mid x = (x_1,x_2) \in M,\ u \in S\},$$

where $(x,0)^{\mathrm{T}}$ denotes the column vector with entries $x_1, x_2, 0$. This is the set in $\mathbb{R}^3$ described by rotating $M \times \{0\}$ about the x_2-axis.

4 pt (a) If $\varphi = (\varphi_1, \varphi_2) :]a,b[\to \mathbb{R}^2$ is an embedding onto an open subset of M, show that for every $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta < \alpha + 2\pi$ the map $\psi :]a,b[\times]\alpha,\beta[\to \mathbb{R}^3$, defined by

$$\psi(s,t) := R_{\sigma(t)}[(\varphi(s), 0)^{\mathrm{T}}]$$

is C^1 and satisfies

$$\|D_1\psi(s,t) \times D_2\psi(s,t)\| = \varphi_1(s)\|\varphi'(s)\|.$$

Show that ψ is a C^1-immersion.

In the following you may use without proof that ψ is actually an embedding onto an open subset of N, and that every point of N is contained in the image of an embedding of this type.

2 pt (b) Show that N is a compact 2-dimensional submanifold of $\mathbb{R}^3$.

4 pt (c) If φ and ψ are as in (a) and if $f : N \to \mathbb{R}$ is a continuous function with support contained in $\psi(]a,b[\times]\alpha,\beta[)$, show that

$$\int_N f(y)\, d_2y = \int_S \int_M x_1 f(R_u[(x,0)^{\mathrm{T}}])\, d_1x\, d_1u.$$

2 pt (d) Show that $\mathrm{vol}_2(N) = 2\pi \int_M x_1\, d_1x$.

12 pt total **Exercise 4.** Let $v : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ be the vector field defined by $v(x) = -x/\|x\|^3$. Let $f \in C^1_c(\mathbb{R}^3)$ have support contained in the open ball $B(0;R)$, $(R > 0)$.

2 pt (a) Show that $\mathrm{div}\, v = 0$ on $\mathbb{R}^3 \setminus \{0\}$.

2 pt (b) Show that $\mathrm{div}(vf) = \langle v, \mathrm{grad} f \rangle$ on $\mathbb{R}^3 \setminus \{0\}$.

In the sequel, you may use the following facts without proof. For $0 < \delta < R$ the set $\Omega_\delta := B(0;R) \setminus \bar{B}(0;\delta)$ is a bounded domain with C^1-boundary. The function $\langle v, \mathrm{grad} f \rangle$ is continuous on the (compact, Jordan measurable) closure of Ω_δ, hence Riemann-integrable over Ω_δ. The associated Riemann-integral is denoted by

$$I_\delta := \int_{B(0;R)\setminus\bar{B}(0;\delta)} \langle v(x), \mathrm{grad} f(x) \rangle\, dx.$$

5 pt (c) Show that for every $0 < \delta < R$ we have $I_\delta = \delta^{-2} \int_{\partial B(0;\delta)} f(y) d_2y$.

3 pt (d) Show that $\lim_{\delta \downarrow 0} I_\delta = 4\pi f(0)$. Hint: consider the integral of $f - f(0)$ over $\partial B(0;\delta)$.