Stochastic Processes (WISB 362) - Final Exam

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This exam consists of 4 questions and 1 bonus question. Please motivate all your answers well. Good luck!

Question 1 [5 points]

Jimmy is bored and is rolling a pair of standard dice (i.e., each die has 6 faces) until a double six appears. Let N be the total number of rolls and let X be the number of times that the sum of the two faces shown is 3. Compute $\mathbb{E}(X|N)$.

Question 2 [20 points]

Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{10} & \frac{2}{10} & \frac{4}{10} & 0 & \frac{1}{10} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \alpha & 0 & \frac{1}{2} - \alpha & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

where $0 \leq \alpha \leq \frac{1}{2}$.

(a) (4 points) Draw a transition diagram, determine the communicating classes, and specify which are closed.

(b) (3 points) Determine for each state whether it is recurrent or transient.

(c) (3 points) Determine the period of each state.

(d) (4 points) Suppose that $\alpha = 0$ and let $A = \{1, 3\}$. Compute the hitting probabilities h_i^A for $i = 1, 2, 3, 4, 5$.

(e) (6 points) Suppose that $\alpha = \frac{1}{2}$. Compute $\lim_{n\to\infty} p_{24}(n)$.

Question 3 [12 points]

Let $(X_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ and fix $n \in \mathbb{N}$. In the lecture we derived the probability density function of J_n, the time of the n-th jump of $(X_t)_{t\geq 0}$. In this question we will derive this probability density function in a different way.

(a) (8 points) Prove that for any $t > 0$

$$\mathbb{P}(t < J_n < t + h) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} h + o(h)$$

as $h \downarrow 0$.

Hint: interpret the event $\{t < J_n < t + h\}$ in terms of the process $(X_t)_{t\geq 0}$.

(b) (4 points) Use part (a) to find the probability density function of J_n.

Question 4 [8 points]

Let $(X_n)_{n\geq 0}$ be a symmetric random walk on $\mathbb{Z}$ and suppose that $X_0 = 0$. Fix $a \in \mathbb{Z}$ and let $T_a = \inf\{n \geq 0 \ : \ X_n = a\}$. For every $n \geq 0$ define the random variable Y_n by

$$Y_n = \begin{cases} X_n & \text{if } n < T_a, \\ 2a - X_n & \text{if } n \geq T_a. \end{cases}$$

Show that $(Y_n)_{n\geq 0}$ is a symmetric random walk on $\mathbb{Z}$.

Hint: analyze the increments of the process.

Bonus question [6 points (bonus)]

Yasmine is studying the Markov Chain Monte Carlo method. To understand it better, she is trying to figure out how it could be used to simulate the random vector (X, Y, Z) with probability mass function

$$\mathbb{P}((X, Y, Z) = (x, y, z)) = \begin{cases} \frac{1}{4} & \text{if } (x, y, z) = (1, 0, 0) \\ \frac{1}{4} & \text{if } (x, y, z) = (0, 1, 0) \\ \frac{1}{2} & \text{if } (x, y, z) = (1, 1, 1). \end{cases}$$

(a) (3 points) At each stage of the MCMC algorithm, Jasmine generates a proposal for the new state by picking a state uniformly at random from the state space. Write down the transition matrix P of the Markov chain that is simulated by the algorithm.

(b) (3 points) Yasmine's algorithm outputs the following sequence of states:

$$(0, 1, 0), (0, 1, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 0, 0), (1, 1, 1), (0, 1, 0).$$

She would like to use this output to make an estimate of $\mathbb{E}(X + Y + Z)$. How would you advise her to make the estimate? Motivate your answer.