Stochastic Processes (WISB 362) - Re-take Exam

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July 14, 2020, 13:30-16:30

This exam consists of 5 questions and 1 bonus question. Please motivate all your answers well. Good luck!

Question 1 [6 points]
Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{10} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{10} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Suppose that we know that $X_0 = 2$. Write down an efficient algorithm to simulate the state X_1 of the Markov chain at time 1.

Question 2 [12 points]
Consider a Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (2 points) Draw a transition diagram, determine the communicating classes, and specify which are closed.

(b) (2 points) Determine for each state whether it is recurrent or transient.

(c) (2 points) Determine the period of each state.

(d) (6 points) Compute $\lim_{n\to\infty} p_{12}(2n)$.

Hint: Exercise 4 of exercise set 4 may be helpful.

Question 3 [8 points]
Jeffrey works during the night shift at a call center. During the night, he is alone on duty. Jeffrey is idle at 03:00 and decides to pick up a coffee. He returns to his desk at 03:30 and finds that there is one caller waiting. What is the probability that the caller has been waiting for more than 10 minutes? You may assume that calls arrive at the call center according to a Poisson process with a rate of one per hour during the night. In addition, you may assume that callers keep waiting (with infinite patience) until they talk to Jeffrey.

Question 4 [8 points]
Let $(X_n)_{n\geq 0}$ be a Markov process. Assume that the process does not have any absorbing states. Define a sequence $(S_m)_{m\geq 0}$ of $\mathbb{N} \cup \{0\}$-valued random variables by $S_0 = 0$ and

$$S_{m+1} = \inf\{n \geq S_m \ : \ X_n \neq X_{S_m}\}.$$

(a) (3 points) Show that $(S_m)_{m\geq 0}$ is a sequence of stopping times.

(b) (5 points) Show that the process $(Z_m)_{m\geq 0}$ defined by $Z_m = X_{S_m}$ is a Markov chain.

Question 5 [11 points]

A frog is hopping from side to side on a straight line. The legs of the frog are unequally strong. As a consequence, with each hop the frog moves 50 cm to the right with probability p and 25 cm to the left otherwise.

(a) (5 points) Model the frog's movements as a Markov chain with state space $\mathbb{Z}$ and determine the probability that the frog returns to its starting position after hopping n times (where n is any natural number).

(b) (6 points) Under what condition on p is the frog certain to return to its starting point? Give a full proof of your answer.

Bonus question [6 points (bonus)]

Jimmy is bored again and is rolling a pair of standard dice (i.e., each die has 6 faces) until a double one appears. Let N be the total number of rolls and let X be the number of times that the sum of the two faces shown is 5. Calculate the probability generating function of X.

Hint: start by calculating the probability generating function conditionally on the value of N.