Universiteit Utrecht
Blok 2, 2019/20
Dr. Carolin Kreisbeck
Dominik Engl (TA)
Jelle Soons (TA)

Mathematical Modeling

WISB357

Final Exam

Name:

Student number:

Signature:

Date: Tuesday, January 28, 2020

Time: 13:30 - 16:30 (3 hours)

Room: Educatorium GAMMA

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- Use a *separate* sheet for each problem.
- The use of textbooks, calculators, cell phones, etc. is *not* allowed.
- Each student is permitted one sheet (format A5, single-sided) of hand-written notes.
- Make sure that your answers are *readable*, *understandable*, and *well justified.*
- Problems marked with * are bonus questions.

Total points: 60 (including 3 bonus points)

Score:

1	2	3	4	Σ

Grade:

Problem 1 (Non-dimensionalization)

Consider the following initial-value problem describing the growth of a population under restricted resources:

$$\begin{cases} x'(t) = q\kappa x(t) - qx^2(t) \quad \text{for } t > 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x(t)$ stands for the size of the population at time t, $q > 0$ is a given growth factor, $x_0 > 0$ is the size of the initial population, and $\kappa > 0$ the maximum capacity. Suppose that $\kappa \gg x_0$.

a) Non-dimensionalize (1) to obtain a problem of the form

$$\begin{cases} y'(\tau) = y(\tau) - \varepsilon y^2(\tau) \quad \text{for } \tau > 0, \\ y(0) = 1, \end{cases} \tag{2}$$

with a parameter $\varepsilon > 0$. Determine ε explicitly in terms of the system parameters. Explain why this non-dimensionalization is well-suited for the given model.

6p

b) Find an alternative way of non-dimensionalizing (1) and discuss the suitability of the resulting system.

3p

c) Determine the formal asymptotic expansion of the solution to (2) up to the first order in ε.

5p

Problem 2 (Competition model)

The system of ordinary differential equations

$$\begin{cases} x' = x(1 - x - y), \\ y' = y(1 - 2x), \end{cases} \tag{3}$$

models the competition between two species of size $x \geq 0$ and $y \geq 0$.

a) Calculate all stationary points and determine their stability behavior. 7p

b) Sketch a phase portrait of (3), including stationary points, isoclines and areas of monotonicity.

4p

c) Under consideration of b), give a heuristic description of the asymptotic behavior of solutions with initial values (x_0, y_0) that satisfy $x_0 > y_0$. Interpret your observations in the context of the model.

Hint: You may use without proof that the line $\{(x, y) \in [0, \infty)^2 : x = y\}$ consists of orbits of (3).

4p

CONTINUATION ON NEXT PAGE

Problem 3 (Orthogonal projection)

We are searching for the shortest distance between a point and an infinite straight line.

a) Explain why the situation above can be described by the following variational problem:

$$\text{Minimize } \mathcal{J}(\gamma) = \int_0^{L_\gamma} |\gamma'(t)| \, \mathrm{d}t \quad \text{ for } \gamma \in \mathcal{A}, \tag{4}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$ and

$$\mathcal{A} = \{\gamma \in C^2([0,L];\mathbb{R}^2) : \gamma(0) = a, \gamma(L) = \beta b \text{ for some } \beta \in \mathbb{R} \text{ and } L > 0\}$$

with given vectors $a, b \in \mathbb{R}^2$; for any $\gamma \in \mathcal{A}$, $L_\gamma > 0$ stands for the right boundary of the interval of definition of γ. What are the assumptions entering this problem formulation? 3p

b) Identify the (maximal) set of admissible variation directions for any $\gamma \in \mathcal{A}$. 3p

One can show that the reparametrization by arc-length of any solution to the minimization problem (4) generates again a minimizer of $\mathcal{J}$ in $\mathcal{A}$.

c) Suppose that $\bar{\gamma}$ solves (4) and has the property that $|\bar{\gamma}'| = 1$. Determine the first variation of $\mathcal{J}$ in $\bar{\gamma}$. Prove then that

$$\bar{\gamma}'' = 0 \quad \text{and} \quad \bar{\gamma}'(L_{\bar{\gamma}}) \cdot b = 0, \tag{5}$$

where $\cdot$ stands for the standard inner product on $\mathbb{R}^2$. 8p

d)* Use the necessary conditions in (5) to calculate $\bar{\gamma}$ explicitly for $a = 2e_1 + e_2$ and $b = e_2$, where e_1, e_2 are the standard unit vectors in $\mathbb{R}^2$. Give an interpretation of your result. 3p

Problem 4 (Continuum mechanics)

a) Derive the continuity equation from the mass conservation in Euler coordinates. Make sure to introduce all the relevant quantities and to explain their meaning. 6p

b) The flow of an isothermal, incompressible, non-viscous fluid in a closed container can be modeled via the system

$$\begin{cases} \nabla \cdot v = 0 & \text{in } (0,\infty) \times \Omega, \\ \partial_t v + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + f & \text{in } (0,\infty) \times \Omega, \end{cases}$$

and the boundary condition

$$v = 0 \quad \text{on } (0,\infty) \times \partial\Omega.$$

Here, $\Omega \subset \mathbb{R}^3$ corresponds to the container, $\rho > 0$ is the (constant) mass density, and f describes external forces. Note that the differential operators ∇ and $\nabla \cdot$ refer only to the spatial variable $x \in \Omega$.

Let $K(t) = \frac{\rho}{2} \int_\Omega |v(t,x)|^2 \, \mathrm{d}x$ be the kinetic energy of the flow at time $t > 0$. Show that

$$K'(t) = \rho \int_\Omega f(t,x) \cdot v(t,x) \, \mathrm{d}x \tag{6}$$

for all $t > 0$. 8p

Sketch of suggested solutions

Please email errors and/or suggestions to c.kreisbeck@uu.nl.

Problem 1

a) We set $x(t) = \bar{x}y(\frac{t}{\bar{t}})$ and plug it into (1) to obtain

$$\frac{\bar{x}}{\bar{t}}y' = q\kappa\bar{x}y - q\bar{x}^2y^2, \quad y(0) = \frac{x_0}{\bar{x}}. \tag{7}$$

After dividing the equation by $\frac{\bar{x}}{\bar{t}}$, (7) becomes

$$y' = q\kappa\bar{t}y - q\bar{x}\bar{t}y^2, \quad y(0) = \frac{x_0}{\bar{x}}. \tag{8}$$

We choose $\bar{x} = x_0$ and $\bar{t} = \frac{1}{q\kappa}$, which yields a problem of the form (2) with $\varepsilon = \frac{x_0}{\kappa}$. If $\kappa \gg x_0$, then $\varepsilon \ll 1$. Neglecting the term involving ε in (2) leads to

$$y' = y, \quad y(0) = 1,$$

which is a model for unrestricted population growth. This is a suitable description of the situation where the maximal capacity of a population is much larger than the initial population, since the species may grow approximately without limitations.

b) *Option 1:* Choose $\bar{t} = \frac{1}{q\kappa}$ and $\bar{x} = \kappa$, then

$$y' = y - y^2, \quad y(0) = \varepsilon$$

with $\varepsilon = \frac{x_0}{\kappa}$. If we set $\varepsilon = 0$, the problem has only the trivial solution $y \equiv 0$.

Option 2: With $\bar{x} = x_0$ and $\bar{t} = \frac{1}{qx_0}$, we obtain

$$y' = \frac{1}{\varepsilon}y - y^2, \quad y(0) = 1$$

with $\varepsilon = \frac{x_0}{\kappa}$. If we multiply the equation with ε, this gives

$$\varepsilon y' = y - \varepsilon y^2, \quad y(0) = 1.$$

The corresponding with $\varepsilon = 0$ is

$$y = 0, \quad y(0) = 1,$$

which does not have a solution.

In contrast to a), these dimensionless initial value problems are not suitable for the case when the capacity is much larger than the initial population.

c) Inserting the ansatz $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + ...$ into (2) and sorting the terms by the orders of ε gives

$$\begin{aligned} y_0' &= y_0, \quad y_0(0) = 1, \\ y_1' &= y_1 - y_0^2, \quad y_1(0) = 0, \\ &\vdots \end{aligned}$$

We solve these equations iteratively. Via variation of constants one finds that

$$y_0(t) = e^t \quad \text{and} \quad y_1(t) = -e^{2t} + e^t$$

for $t > 0$.

Problem 2

a) The stationary points can be calculated by solving the system

$$\begin{cases} 0 = x(1-x-y), \\ 0 = y(1-2x), \end{cases}$$

which has the solutions $(0,0), (1,0), (\frac{1}{2}, \frac{1}{2})$. To determine their linearized stability, we linearize the vector field $f(x,y) = (x(1-x-y), y(1-2x))^T$ at the stationary points. The derivative of f is given by

$$Df(x,y) = \begin{pmatrix} 1-2x-y & -x \\ -2y & 1-2x \end{pmatrix}.$$

Evaluating at $(0,0)$ gives $Df(0,0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which has eigenvalue $1 > 0$ with multiplicity 2. This implies that the origin is linearly unstable. By the principle of linearized stability, we can therefore conclude that $(0,0)$ is unstable.

At $(1,0)$, we obtain $Df(1,0) = \left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$, which has the eigenvalue $-1 < 0$ with multiplicity 2. Consequently, $(1,0)$ is linearly asymptotically stable, which yields asymptotic stability for $(1,0)$.

Finally, at $(\frac{1}{2}, \frac{1}{2})$, we get $Df(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}\left(\begin{smallmatrix} -1 & -1 \\ -2 & 0 \end{smallmatrix}\right)$, whose characteristic polynomial is a multiple of $\lambda^2 + \frac{1}{2}\lambda - \frac{1}{2}$. Since the zeroes of the latter are given by $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -1$, the point $(\frac{1}{2}, \frac{1}{2})$ is (linearly) unstable.

b) Observe that x' vanishes on

$$\{(x,y) \in [0,\infty)^2 : y = 1-x\} \cup \{(x,y) \in [0,\infty)^2 : x = 0\}$$

and that y' vanishes on the set

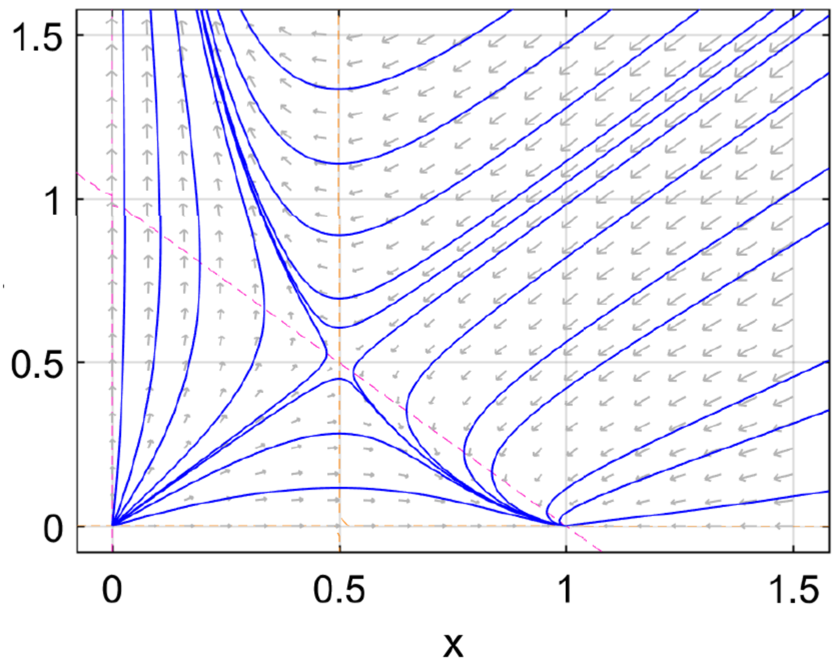
$$\{(x,y) \in [0,\infty)^2 : x = \tfrac{1}{2}\} \cup \{(x,y) \in [0,\infty)^2 : y = 0\}.$$

These four lines describe the isolines of the system (3). Moreover, we obtain the following areas of monotonicity:

$$\begin{aligned} D_{--} &= \{(x,y) \in [0,\infty)^2 : y > 1-x\} \cap \{(x,y) \in [0,\infty)^2 : x > \tfrac{1}{2}\}; \\ D_{-+} &= \{(x,y) \in [0,\infty)^2 : y > 1-x\} \cap \{(x,y) \in [0,\infty)^2 : x < \tfrac{1}{2}\}; \\ D_{++} &= \{(x,y) \in [0,\infty)^2 : y < 1-x\} \cap \{(x,y) \in [0,\infty)^2 : x < \tfrac{1}{2}\}; \\ D_{+-} &= \{(x,y) \in [0,\infty)^2 : y < 1-x\} \cap \{(x,y) \in [0,\infty)^2 : x > \tfrac{1}{2}\}. \end{aligned}$$

c) Exploiting the structure of the monotonicity areas and the resulting insight into the direction of the vector field $f = (f_1, f_2)^T$ on the isoclines, we show that every solution to (3) with initial value $x_0 > y_0$ converges to $(1,0)$. We argue separately in the following three cases:

i) Suppose $y_0 < 1-x_0$ and $x_0 > \frac{1}{2}$, or equivalently, $(x_0, y_0) \in D_{+-}$. Since $\{(x,y) \in [0,\infty)^2 : y = 0\}$ is an isocline on which $f_1 > 0$ and $f_2 = 0$, and $\{(x,y) \in [0,\infty)^2 : x > y \text{ and } y = 1-x\}$ is an



isocline along which $f_2 < 0$ and $f_1 = 0$, the solution stays inside D_{+-} for all times. Then, due to the asymptotic stability of the stationary point $(1, 0)$, we conclude that $x(t) \to (1, 0)$ for $t \to \infty$.

ii) Let $y_0 > 1 - x_0$ and $x_0 > \frac{1}{2}$. Then $(x_0, y_0) \in D_{--}$ and thus, the solution converges to the stationary point $(1, 0)$ or intersects the isocline $\{(x, y) \in [0, \infty)^2 : y = 1 - x\}$ at some point $t_0 > 0$ in time. In the latter case, the same argument as in *i*) shows that $(x(t), y(t)) \in D_{+-}$ for all times $t > t_0$, and that the solution converges to $(1, 0)$ also in this situation.

iii) If $y_0 < 1 - x_0$ and $x_0 < \frac{1}{2}$, then $(x_0, y_0) \in D_{++}$. Since $\{(x, y) \in [0, \infty)^2 : x = y\}$ consists of orbits, the solution (x, y) cannot intersect this line due to the local uniqueness of solutions of (3). Therefore, there exists $t_0 > 0$ such that $(x(t_0), y(t_0)) \in \{(x, y) \in [0, \infty)^2 : x = \frac{1}{2}\}$ and $(x(t), y(t)) \in D_{+-}$ for all $t > t_0$ according to *i*). Analogously, we conclude that the solution converges to $(1, 0)$.

The interpretation is that in the long-term, species x eliminates the species y, while approaching its maximal capacity.

Problem 3

a) In this planar model, $a \in \mathbb{R}^2$ describes a point and the span of $b \in \mathbb{R}^2$ a infinite straight line; one can suppose without loss of generality that the origin lies on the line. Assuming enough regularity, precisely the existence and continuity of first and second derivatives, we consider parametrized curves with starting point a and end point on the line, that is, with end point βb for some $\beta \in \mathbb{R}$; indeed, every element of the line can be expressed as a multiple of b. To find the distance from the point to the line, we search among those curve for the one with minimal length, which translates into the task of minimizing the functional $\mathcal{J}$, which describes the lengths of curves.

Remark: Note that this is a free boundary value problem.

b) Fix $\gamma \in \mathcal{A}$. We observe that $\gamma + \varepsilon\sigma \in \mathcal{A}$ for $\varepsilon > 0$ if and only if $\sigma \in C^2([0, L_\gamma]; \mathbb{R}^2)$ with $\sigma(0) = 0$ and $\sigma(L_\gamma) = \hat{\beta} b$ for any $\hat{\beta} \in \mathbb{R}$. In particular, $\sigma \in \mathcal{A}$ with $L_\sigma = L_\gamma$.

c) For any σ as in b), we obtain under consideration of $|\bar\gamma'| = 1$ that

$$\begin{aligned}\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{J}(\bar\gamma + \varepsilon\sigma) &= \frac{d}{d\varepsilon}|_{\varepsilon=0}\int_0^{L_{\bar\gamma}} |\bar\gamma'(t) + \varepsilon\sigma'(t)| \text{ d}t = \frac{d}{d\varepsilon}|_{\varepsilon=0}\int_0^{L_{\bar\gamma}} \sqrt{|\bar\gamma'|^2 + \varepsilon^2|\sigma'|^2 + 2\varepsilon\gamma'\cdot\sigma'} \text{ d}t \\ &= \int_0^{L_{\bar\gamma}} \frac{\bar\gamma'}{|\bar\gamma'|}\cdot\sigma' \text{ d}t = \int_0^{L_{\bar\gamma}} \bar\gamma'\cdot\sigma' \text{ d}t = -\int_0^{L_{\bar\gamma}} \bar\gamma''\cdot\sigma \text{ d}t + \sigma(L_{\bar\gamma})\cdot\bar\gamma'(L_{\bar\gamma})\end{aligned}$$

In the last step, we have used integration by parts together with the zero boundary condition of σ on the left boundary of the interval and the regularity of $\bar\gamma$.

Since $\bar\gamma$ is a minimizer of $\mathcal{J}$ in $\mathcal{A}$, it is also a critical point for $\mathcal{J}$, meaning that

$$-\int_0^{L_{\bar\gamma}} \bar\gamma''\cdot\sigma \text{ d}t + \bar\gamma'(L_{\bar\gamma})\cdot\sigma(L_{\bar\gamma}) = 0. \tag{9}$$

Choosing σ in (9) with $\sigma(L_{\bar\gamma}) = 0$ and $\sigma_2 = 0$ yields in view of the fundamental lemma in the calculus of variations that $\bar\gamma_1'' = 0$ on $(0, L_{\bar\gamma})$, and analogously, it follows that $\bar\gamma_2'' = 0$ on $(0, L_{\bar\gamma})$. Hence, $\bar\gamma'' = 0$ on $(0, L_{\bar\gamma})$, and (9) turns into

$$\sigma(L_{\bar\gamma})\cdot\bar\gamma'(L_{\bar\gamma}) = 0.$$

If we now take σ with $\sigma(L_{\bar\gamma}) = b$, this yields $\bar\gamma'(L_{\bar\gamma})\cdot b = 0$ as stated.

d) We infer from c) that $\bar\gamma'$ is constant, and thus, $\bar\gamma$ is affine, i.e., $\bar\gamma(t) = \bar a + t\bar b$ for $t \in (0, L)$ with $\bar a, \bar b \in \mathbb{R}^2$. Along with the boundary conditions resulting from $\bar\gamma \in \mathcal{A}$ and the specific choices of a and b, one obtains that

$$\bar\gamma(t) = -\frac{2t}{L_{\bar\gamma}}e_1 + 2e_1 + e_2 \quad \text{for } t \in (0, L_{\bar\gamma}).$$

Since $\bar\gamma$ is parametrized by arc length, we infer that $L_{\bar\gamma} = 2$. The end point of $\bar\gamma$ is the orthogonal projection of a onto the one-dimensional space spanned by b, and the distance of a from this space is $L_{\bar\gamma} = 2$.

Problem 4

a) We introduce the following relevant quantities:

- $\Omega \subset \mathbb{R}^d$ reference configuration
- $x(t, X)$ for $X \in \Omega$ and $t > 0$ position of the material point X at time t
- $\Omega'(t) = \{x(t, X) : X \in \Omega'\}$ with $\Omega' \subset \Omega$ volume transported by the evolution of mass points
- $v(t, x)$ velocity field in Euler coordinates
- $\rho(t, x)$ mass density in Euler coordinates

Mass conservation is given by

$$\frac{d}{dt}\int_{\Omega'(t)} \rho(t, x) \text{ d}x = 0,$$

which, by the Reynolds transport theorem, can be equivalently rewritten as

$$\int_{\Omega'(t)} \partial_t\rho(t, x) + \nabla\cdot(\rho(t, x)v(t, x)) \text{ d}x = 0.$$

As a consequence of the localization lemma, we conclude that the integrand of the previous identity needs to vanish pointwise for every t, and hence,

$$\partial_t \rho + \nabla \cdot (\rho v) = 0.$$

This is the continuity equation.

b) Considering the given system

$$\begin{cases} \nabla \cdot v = 0 & \text{in } (0,\infty) \times \Omega, \\ \partial_t v + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + f & \text{in } (0,\infty) \times \Omega, \end{cases} \tag{10}$$

the strategy is to (scalar) multiply the second equation in (10) with v, integrate over Ω, and use integration by parts, exploiting the given boundary data and the first equation in (10).
In the following calculations, $n \in \mathbb{R}^3$ denotes the outer unit normal of $\partial\Omega$.

By $\partial_t v \cdot v = \frac{1}{2}\partial_t |v|^2$, the first term on the left-hand side becomes

$$\int_\Omega \rho \partial_t v(t,x) \cdot v(t,x) \, \mathrm{d}x = \frac{\rho}{2} \int_\Omega \partial_t |v(t,x)|^2 \, \mathrm{d}x = \frac{d}{dt}\Big(\frac{\rho}{2} \int_\Omega |v(t,x)|^2 \, \mathrm{d}x \Big)$$

for every $t > 0$. For the second term, we show that $\int_\Omega (v \cdot \nabla) v \cdot v \, \mathrm{d}x = 0$; indeed, under consideration of the boundary values of v,

$$\begin{aligned} \int_\Omega (v \cdot \nabla) v \cdot v \, \mathrm{d}x &= \sum_{i,j=1}^3 \int_\Omega v_i v_j \partial_j v_i \, \mathrm{d}x = \sum_{i,j=1}^3 \Big[- \int_\Omega \partial_j (v_i v_j) v_i \, \mathrm{d}x + \int_{\partial\Omega} v_i v_j v_i n_j \, \mathrm{d}s_x \Big] \\ &= - \sum_{i,j=1}^3 \int_\Omega v_j (\partial_j v_i) v_i + (\partial_j v_j) v_i^2 \, \mathrm{d}x = - \int_\Omega (v \cdot \nabla) v \cdot v + |v|^2 \nabla \cdot v \, \mathrm{d}x \\ &= - \int_\Omega (v \cdot \nabla) v \cdot v \, \mathrm{d}x, \end{aligned}$$

where, in the last step, we have used that v is divergence free (with respect to the spatial variables) by the first equation in (10). Finally, the first term on the right-hand side vanishes in view of $\nabla \cdot v = 0$, precisely,

$$\int_\Omega \nabla p \cdot v \, \mathrm{d}x = - \int_\Omega p \nabla \cdot v \, \mathrm{d}x + \int_{\partial\Omega} p v \cdot n \, \mathrm{d}s_x = 0.$$

Summing up, we have shown that

$$\frac{d}{dt} \frac{\rho}{2} \int_\Omega (|v(t,x)|^2) \, \mathrm{d}x = \rho \int_\Omega f(t,x) \cdot v(t,x) \, \mathrm{d}x$$

for all times $t > 0$ and by definition of K, it follows that

$$K'(t) = \rho \int_\Omega f(t,x) \cdot v(t,x) \, \mathrm{d}x,$$

as stated.