Mathematical Modeling

WISB357

Universiteit Utrecht
Blok 2, 2019/20
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Final Exam

Name:

Student number:

Signature:

Date: Tuesday, January 28, 2020

Time: 13:30 - 16:30 (3 hours)

Room: Educatorium GAMMA

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- Use a *separate* sheet for each problem.
- The use of textbooks, calculators, cell phones, etc. is *not* allowed.
- Each student is permitted one sheet (format A5, single-sided) of hand-written notes.
- Make sure that your answers are *readable*, *understandable*, and *well justified.*
- Problems marked with * are bonus questions.

Total points: 60 (including 3 bonus points)

Score:

1	2	3	4	Σ

Grade:

Problem 1 (Non-dimensionalization)

Consider the following initial-value problem describing the growth of a population under restricted resources:

$$\begin{cases} x'(t) = q\kappa x(t) - q x^2(t) \quad \text{for } t > 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x(t)$ stands for the size of the population at time t, $q > 0$ is a given growth factor, $x_0 > 0$ is the size of the initial population, and $\kappa > 0$ the maximum capacity. Suppose that $\kappa \gg x_0$.

a) Non-dimensionalize (1) to obtain a problem of the form

$$\begin{cases} y'(\tau) = y(\tau) - \varepsilon y^2(\tau) \quad \text{for } \tau > 0, \\ y(0) = 1, \end{cases} \tag{2}$$

with a parameter $\varepsilon > 0$. Determine ε explicitly in terms of the system parameters. Explain why this non-dimensionalization is well-suited for the given model.

6p

b) Find an alternative way of non-dimensionalizing (1) and discuss the suitability of the resulting system.

3p

c) Determine the formal asymptotic expansion of the solution to (2) up to the first order in ε.

5p

Problem 2 (Competition model)

The system of ordinary differential equations

$$\begin{cases} x' = x(1 - x - y), \\ y' = y(1 - 2x), \end{cases} \tag{3}$$

models the competition between two species of size $x \geq 0$ and $y \geq 0$.

a) Calculate all stationary points and determine their stability behavior.

7p

b) Sketch a phase portrait of (3), including stationary points, isoclines and areas of monotonicity.

4p

c) Under consideration of b), give a heuristic description of the asymptotic behavior of solutions with initial values (x_0, y_0) that satisfy $x_0 > y_0$. Interpret your observations in the context of the model.

Hint: You may use without proof that the line $\{(x, y) \in [0, \infty)^2 : x = y\}$ consists of orbits of (3).

4p

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Problem 3 (Orthogonal projection)

We are searching for the shortest distance between a point and an infinite straight line.

a) Explain why the situation above can be described by the following variational problem:

$$\text{Minimize } \mathcal{J}(\gamma) = \int_0^{L_\gamma} |\gamma'(t)| \,\mathrm{d}t \quad \text{for } \gamma \in \mathcal{A}, \tag{4}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$ and

$$\mathcal{A} = \{\gamma \in C^2([0,L];\mathbb{R}^2) : \gamma(0) = a, \gamma(L) = \beta b \text{ for some } \beta \in \mathbb{R} \text{ and } L > 0\}$$

with given vectors $a, b \in \mathbb{R}^2$; for any $\gamma \in \mathcal{A}$, $L_\gamma > 0$ stands for the right boundary of the interval of definition of γ. What are the assumptions entering this problem formulation? 3p

b) Identify the (maximal) set of admissible variation directions for any $\gamma \in \mathcal{A}$. 3p

One can show that the reparametrization by arc-length of any solution to the minimization problem (4) generates again a minimizer of $\mathcal{J}$ in $\mathcal{A}$.

c) Suppose that $\bar{\gamma}$ solves (4) and has the property that $|\bar{\gamma}'| = 1$. Determine the first variation of $\mathcal{J}$ in $\bar{\gamma}$. Prove then that

$$\bar{\gamma}'' = 0 \quad \text{and} \quad \bar{\gamma}'(L_{\bar{\gamma}}) \cdot b = 0, \tag{5}$$

where $\cdot$ stands for the standard inner product on $\mathbb{R}^2$. 8p

d)* Use the necessary conditions in (5) to calculate $\bar{\gamma}$ explicitly for $a = 2e_1 + e_2$ and $b = e_2$, where e_1, e_2 are the standard unit vectors in $\mathbb{R}^2$. Give an interpretation of your result. 3p

Problem 4 (Continuum mechanics)

a) Derive the continuity equation from the mass conservation in Euler coordinates. Make sure to introduce all the relevant quantities and to explain their meaning. 6p

b) The flow of an isothermal, incompressible, non-viscous fluid in a closed container can be modeled via the system

$$\begin{cases} \nabla \cdot v = 0 & \text{in } (0,\infty) \times \Omega, \\ \partial_t v + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + f & \text{in } (0,\infty) \times \Omega, \end{cases}$$

and the boundary condition

$$v = 0 \quad \text{on } (0,\infty) \times \partial\Omega.$$

Here, $\Omega \subset \mathbb{R}^3$ corresponds to the container, $\rho > 0$ is the (constant) mass density, and f describes external forces. Note that the differential operators ∇ and $\nabla\cdot$ refer only to the spatial variable $x \in \Omega$.

Let $K(t) = \frac{\rho}{2} \int_\Omega |v(t,x)|^2 \,\mathrm{d}x$ be the kinetic energy of the flow at time $t > 0$. Show that

$$K'(t) = \rho \int_\Omega f(t,x) \cdot v(t,x) \,\mathrm{d}x \tag{6}$$

for all $t > 0$. 8p