Universiteit Utrecht
Blok 2, 2019/20
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Mathematical Modeling

WISB357

Retake Exam

Name:

Student number:

Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van andere hulpmiddelen dan het cursusboek, material uit het werkcollege en eigen aantekeningen.

Signature:

Date: Tuesday, April 14, 2020

Time: 13:30 - 16:30 (3 hours)

Instructions:

- Write your *name, student number, and problem number* on every page you submit electronically.
- Consulting the internet and using calculators, cell phones, etc. is *not* allowed.
- You are permitted to make use of the course book, material from the exercise sessions, and any hand-written notes.
- Make sure that your answers are *readable*, *understandable*, and *well justified.*
- Problems marked with * are bonus questions.

Total points: 60 (including 3 bonus points)

Score:

1	2	3	4	Σ

Grade:

Problem 1

The following system of ordinary differential equations can be used to model the growth of two competing species in an environment of restricted resources:

$$\begin{cases} S_1' &= r_1 S_1 - q_1 S_1^2 - a S_1 S_2, \\ S_2' &= r_2 S_2 - q_2 S_2^2 - b S_1 S_2, \end{cases} \tag{1}$$

where $r_1, r_2 > 0$ are the growth rates of the species $S_1, S_2 \geq 0$, $q_1, q_2 > 0$ reflect the effects of a maximal capacity for the individual species, and $a, b > 0$ correspond to the rates at which the two species eliminate each other.

a) Non-dimensionalize (1) to obtain a system of the form

$$\begin{cases} x_1' &= x_1 - x_1^2 - d_2 x_1 x_2, \\ x_2' &= \rho x_2 - x_2^2 - d_1 x_1 x_2, \end{cases} \tag{2}$$

with $\rho, d_1, d_2 > 0$, and determine these three quantities explicitly in terms of the parameters r_1, r_2, q_1, q_2, a, b. Give an interpretation of ρ in context of the model. 6p

Henceforth, we assume that $d_2 = 1$ and $d_1 = \frac{1}{2}$.

b) Characterize the set of all $\rho > 0$ such that (2) has an asymptotically stable equilibrium that corresponds to a situation where neither of the two species gets extinct. 7p

c) Sketch a phase portrait of (2), including stationary points, isoclines and areas of monotonicity for $\rho = \frac{3}{4}$. 4p

d) Describe and explain the qualitative difference between the long-term behavior of solutions to (2) in the two regimes $\rho = \frac{3}{4}$ and $\rho = 2$. 4p

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Problem 2

Consider the boundary-value problem

$$\begin{cases} \partial_t u(t,x) - \nabla \cdot (A(x)\nabla u(t,x)) = f(t,x) & \text{for } (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \partial\Omega, \end{cases} \tag{3}$$

with unknown $u : (0,\infty) \times \Omega \to \mathbb{R}$. Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $f : (0,\infty) \times \Omega \to \mathbb{R}^n$, $A : \Omega \to \mathbb{R}^{3\times 3}$ are smooth functions, and A is supposed to satisfy

$$\xi \cdot A(x)\xi \geq C|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and all } x \in \Omega, \tag{4}$$

with a constant $C > 0$ independent of x and ξ. Note that the differential operators ∇ and $\nabla\cdot$ involve only partial derivatives with respect to the spatial variable $x \in \Omega$.

a) Prove that any solution u to (3) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}|u(t,x)|^2 \,\mathrm{d}x + \int_\Omega \nabla u(t,x) \cdot A(x)\nabla u(t,x) \,\mathrm{d}x = \int_\Omega f(t,x)u(t,x) \,\mathrm{d}x \tag{5}$$

for every $t > 0$. 6p

b) Use (5) and (4) to conclude that, for given initial values, the solution u to (3) is unique.

Hint: Show that the difference of two solutions to (3) is again a solution to (3) for a special choice of f. 6p

c) Specify $A : \Omega \to \mathbb{R}^{3\times 3}$ such that the partial differential equation in (3) turns into the heat equation. 3p

d)* Is the heat equation elliptic, hyperbolic or parabolic? Justify your answer! 3p

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Problem 3

Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a smooth, 1-periodic function and let $f : (0,1) \to \mathbb{R}$ be a given smooth function. For $\varepsilon > 0$, we consider the initial-value problem

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\lambda\big(\tfrac{t}{\varepsilon}\big)u'(t)\big) = f(t), \quad u''(0) = \frac{1}{\varepsilon} \tag{6}$$

for $t \in (0,1)$.

a) Suppose that $u_\varepsilon : (0,1) \to \mathbb{R}$ is a solution to (6) of the form $u_\varepsilon(t) = v_\varepsilon(t, \frac{t}{\varepsilon})$ for all $t \in (0,1)$, where $v_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ can be expressed as

$$v_\varepsilon(x_1,x_2) = v_0(x_1,x_2) + \varepsilon v_1(x_1,x_2) + \varepsilon^2 v_2(x_1,x_2) + \mathcal{O}(\varepsilon^3), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Find the initial value problems for the orders $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ in the asymptotic expansion of u_ε. 6p

b) Determine a set $\mathcal{A}$ and a functional $\mathcal{I}$ such that (6) with $\varepsilon = 1$ is the Euler-Lagrange equation (with boundary values) of the variational problem

$$\text{Minimize } \mathcal{I}(u) \text{ for } u \in \mathcal{A}.$$

6p

Problem 4

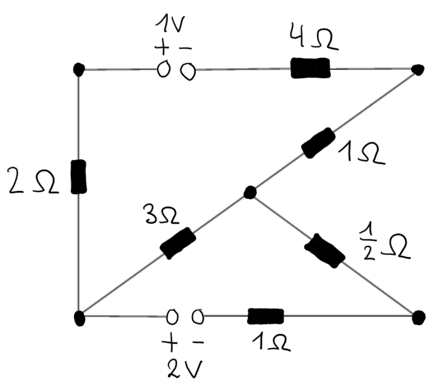
The optimal power within an electrical network can be obtained by solving the optimization problem

$$\text{Minimize } \mathcal{I}(y) = \tfrac{1}{2} y \cdot Ry - b \cdot y \quad \text{for } y \in \mathcal{A} = \{y \in \mathbb{R}^n : B^T y = 0\}, \tag{7}$$

where the matrix of resistances $R \in \mathbb{R}^{n \times n}$ is diagonal and positive definite, $b \in \mathbb{R}^n$ is the voltage source vector, and the incidence matrix $B \in \mathbb{R}^{n \times m}$ describes the network geometry.

a) Interpreting $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ as a functional, identify the (maximal) set of variation directions and compute the first variation of $\mathcal{I}$ at any $y \in \mathbb{R}^n$. Find a necessary condition for solutions to (7). 5p

b) Determine R, B, and b for the following electrical network:



4p

Sketch of suggested solutions

Please email errors and/or suggestions to c.kreisbeck@uu.nl.

Problem 1

a) We set $x_1(\tau) = \frac{1}{\bar{S}_1} S_1(\frac{t}{\bar{t}})$ and $x_2(\tau) = \frac{1}{\bar{S}_2} S_2(\frac{t}{\bar{t}})$ and plug it into (1) to obtain

$$\begin{cases} \frac{\bar{S}_1}{\bar{t}} x_1' &= r_1 \bar{S}_1 x_1 - q_1 \bar{S}_1^2 x_1^2 - a \bar{S}_1 \bar{S}_2 x_1 x_2, \\ \frac{\bar{S}_2}{\bar{t}} x_2' &= r_2 \bar{S}_2 x_2 - q_2 \bar{S}_2^2 x_2^2 - b \bar{S}_1 \bar{S}_2 x_1 x_2. \end{cases} \tag{8}$$

After dividing the equations by $\frac{\bar{S}_1}{\bar{t}}$ and $\frac{\bar{S}_2}{\bar{t}}$, respectively, (8) becomes

$$\begin{cases} x_1' &= r_1 \bar{t} x_1 - q_1 \bar{t} \bar{S}_1 x_1^2 - a \bar{t} \bar{S}_2 x_1 x_2, \\ x_2' &= r_2 \bar{t} x_1 - q_2 \bar{t} \bar{S}_2 x_1^2 - b \bar{t} \bar{S}_1 x_1 x_2. \end{cases} \tag{9}$$

Choosing $\bar{t} = \frac{1}{r_1}$, $\bar{S}_1 = \frac{r_1}{q_1}$, and $\bar{S}_2 = \frac{r_1}{q_2}$ yields a problem of the form (2) with

$$d_1 = \frac{b}{q_1}, \; d_2 = \frac{a}{q_2}, \quad \text{and} \quad \rho = \frac{r_2}{r_1}.$$

The factor ρ describes the ratio of the growth rates of the two competing species. In particular, $\rho \geq 1$ if and only if species S_2 grows at least as fast as species S_1.

b) Observe that x_1' vanishes on

$$\{(x_1, x_2) \in [0, \infty)^2 : x_2 = \tfrac{1}{d_2}(1 - x_1) = 1 - x_1\} \cup \{(x_1, x_2) \in [0, \infty)^2 : x_1 = 0\}$$

and that x_2' vanishes on the set

$$\{(x_1, x_2) \in [0, \infty)^2 : x_2 = \rho - d_1 x_1 = \rho - \tfrac{1}{2} x_1\} \cup \{(x_1, x_2) \in [0, \infty)^2 : x_2 = 0\}.$$

Hence, the stationary points, which correspond to the intersection of these two sets, are

$$(0,0), (1,0), (0,\rho), (2 - 2\rho, 2\rho - 1).$$

Note that the last equilibrium is admissible if and only if $\rho \in [\frac{1}{2}, 1]$, which is what we shall assume from now on. Moreover, in the search for equilibria corresponding to situations where none of the two species are extinct, it suffices to consider $(2 - 2\rho, 2\rho - 1)$ for $\rho \in (\frac{1}{2}, 1)$. We will show that the latter are asymptotically stable stationary points.

Setting $f(x_1, x_2) = \big(x_1(1 - x_1 - x_2), x_2(\rho - x_2 - \frac{1}{2}x_1)\big)^T$, we compute for the Jacobian that

$$Df(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 - x_2 & -x_1 \\ -\frac{1}{2}x_2 & \rho - 2x_2 - \frac{1}{2}x_1 \end{pmatrix};$$

In particular, we find that

$$Df(2 - 2\rho, 2\rho - 1) = \begin{pmatrix} 2\rho - 2 & 2\rho - 2 \\ \frac{1}{2} - \rho & 1 - 2\rho \end{pmatrix}.$$

Since $\mathrm{Tr}[Df(2-2\rho, 2\rho-1)] = -1 < 0$ and $\det Df(2-2\rho, 2\rho-1) = -2(\rho-1)(\rho-\frac{1}{2})$, we conclude $Df(2 - 2\rho, 2\rho - 1)$ has all negative eigenvalues if and only if $\rho \in (\frac{1}{2}, 1)$. Thus, the principle of linearized stability implies that $(2-2\rho, 2\rho-1)$ is (linearly) asymptotically stable if and only if $\rho \in (\frac{1}{2}, 1)$.

c) The areas of monotonicity for such $\rho = \frac{3}{4}$ are given by

$$\begin{aligned}
D_{--} &= \{(x_1,x_2) \in [0,\infty)^2 : x_2 > 1 - x_1\} \cap \{(x_1,x_2) \in [0,\infty)^2 : x_2 > \tfrac{3}{4} - \tfrac{1}{2}x_1\};\\
D_{-+} &= \{(x_1,x_2) \in [0,\infty)^2 : x_2 > 1 - x_1\} \cap \{(x_1,x_2) \in [0,\infty)^2 : x_2 < \tfrac{3}{4} - \tfrac{1}{2}x_1\};\\
D_{++} &= \{(x_1,x_2) \in [0,\infty)^2 : x_2 < 1 - x_1\} \cap \{(x_1,x_2) \in [0,\infty)^2 : x_2 < \tfrac{3}{4} - \tfrac{1}{2}x_1\};\\
D_{+-} &= \{(x_1,x_2) \in [0,\infty)^2 : x_2 < 1 - x_1\} \cap \{(x_1,x_2) \in [0,\infty)^2 : x_2 > \tfrac{3}{4} - \tfrac{1}{2}x_1\}.
\end{aligned}$$

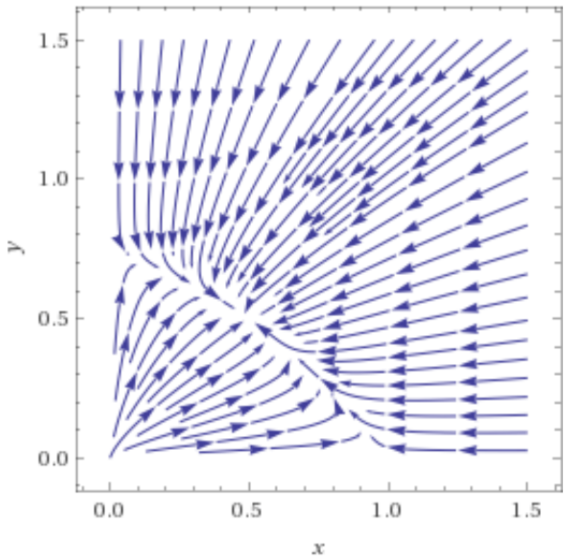


Figure 1: Trajectories of (2) for $\rho = 0.75$

d) If $\rho = 2$, then species S_2 grows double as fast as species S_1 and the only (admissible) stationary points are $(0,0)$, $(1,0)$ and $(0,2)$. At these points, we obtain

$$Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \qquad Df(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & \frac{3}{2} \end{pmatrix},$$

which both have $\frac{3}{2} > 0$ as an eigenvalue. Therefore, $(0,1)$ and $(1,0)$ are (linearly) unstable. On the other hand,

$$Df(0,2) = \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix},$$

which has only negative eigenvalues with $-1, -2$, shows that $(0,2)$ is asymptotically stable by the principle of linearized stability.

Therefore, in contrast to $\rho = \frac{3}{4}$, which, according to c), results in the coexistence of both species at the same size for large times, the species S_2 eliminates S_1 for $\rho = 2$, growing to its maximum capacity (cf. Figure 2).

Problem 2

a) Let $t > 0$ be arbitrary. We multiply the partial differential equation in (3) with u and integrate over Ω to obtain

$$\int_\Omega u(t,x)\partial_t u(t,x) - \nabla \cdot (A(x)\nabla u(t,x))u \,\mathrm{d}x = \int_\Omega u(t,x) f(t,x) \,\mathrm{d}x. \tag{10}$$

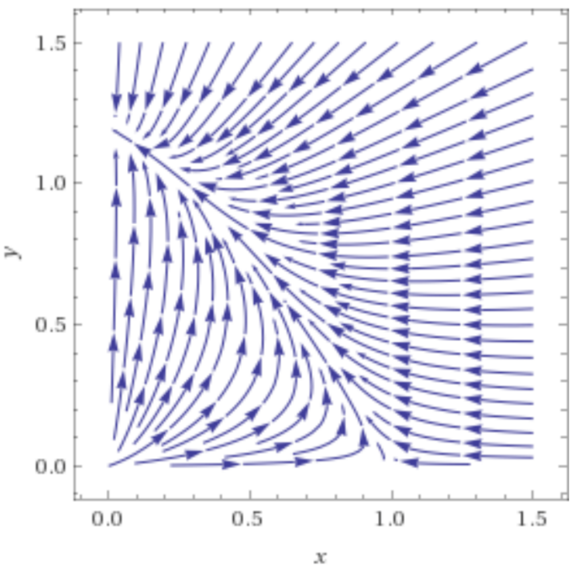


Figure 2: Trajectories of (2) for $\rho = 1.2$

Note that $\partial_t(\frac{1}{2}|u(t,x)|^2) = u(t,x)\partial_t u(t,x)$, from which we deduce that

$$\int_\Omega u(t,x)\partial_t u(t,x)\ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|u(t,x)|^2\ \mathrm{d}x. \tag{11}$$

Moreover, via integration by parts, we get

$$-\int_\Omega \nabla\cdot(A(x)\nabla u(t,x))u\ \mathrm{d}x = \int_\Omega \nabla u(t,x)\cdot A(x)\nabla u(t,x)\ \mathrm{d}x - \int_{\partial\Omega} u(t,x)n(x)\cdot A(x)\nabla u(t,x)\ \mathrm{d}s_x,$$

where n denotes the outer unit normal field of Ω. Exploiting the boundary conditions in (3), we infer that

$$-\int_\Omega \nabla\cdot(A(x)\nabla u(t,x))u\ \mathrm{d}x = \int_\Omega \nabla u(t,x)\cdot A(x)\nabla u(t,x)\ \mathrm{d}x. \tag{12}$$

Finally, joining (10) - (12), yields desired energy equation (5).

b) Let u, v be two solutions of (3) with initial value $v(0,x) = u(0,x) = u_0(x)$ for some $u_0 : \Omega \to \mathbb{R}$. Then, their difference $w := u - v$ satisfies

$$\begin{cases} \partial_t w(t,x) - \nabla\cdot(A(x)\nabla w(t,x)) = 0 & \text{for } (t,x)\in(0,\infty)\times\Omega, \\ w(t,x) = 0 & \text{for } (t,x)\in(0,\infty)\times\partial\Omega, \end{cases}$$

as well as $w(0,x) = 0$ for all $x \in \Omega$. In light of b) for $f = 0$ and $u = w$, we obtain for all $t > 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|w(t,x)|^2\ \mathrm{d}x + \int_\Omega \nabla w(t,x)\cdot A(x)\nabla w(t,x)\ \mathrm{d}x = 0.$$

Integrating this equation over $(0,t)$ for any $t > 0$ and using the initial condition $w(0,x) = 0$ leads to

$$\int_\Omega \frac{1}{2}|w(t,x)|^2\ \mathrm{d}x + \int_0^t\int_\Omega \nabla w(s,x)\cdot A(x)\nabla w(s,x)\ \mathrm{d}x\ \mathrm{d}s = 0.$$

Since by (4),

$$\int_0^t \int_\Omega \nabla w(s,x) \cdot A(x) \nabla w(s,x) \,\mathrm{d}x \,\mathrm{d}s \geq C \int_0^t \int_\Omega |\nabla w(s,x)|^2 \,\mathrm{d}x \,\mathrm{d}s \geq 0,$$

we conclude that

$$\int_\Omega \frac{1}{2} |w(t,x)|^2 \,\mathrm{d}x \leq 0.$$

Thus, $|w(t,\cdot)|$ vanishes everywhere on Ω for any $t > 0$, which means that $u = v$ everywhere on $(0,\infty) \times \Omega$.

c) We choose $A(x) = \lambda \,\mathrm{Id}$, where Id denotes the identity matrix in $\mathbb{R}^{3\times 3}$ and $\lambda > 0$ is the heat coefficient. Then,

$$\nabla \cdot (A(x) \nabla u(t,x)) = \lambda \nabla \cdot \nabla u(t,x) = \lambda \Delta u(t,x),$$

and the differential equation in (3) becomes $\partial_t u - \lambda \Delta u = 0$, which yields the desired heat equation.

d)* The heat equation is parabolic. Indeed, it is a linear second order PDE that can be written in the form

$$0 = \partial_t u - \lambda \Delta u = \partial_t u - \lambda \sum_{j=1}^{3} \partial_{x_j} \partial_{x_j} u = \partial_t u - \lambda \sum_{j,k=1}^{3} \delta_{jk} \partial_{x_j} \partial_{x_k} u = \sum_{j,k=1}^{4} \tilde{a}_{jk} \partial_{y_j} \partial_{y_k} u + \partial_{y_1} u,$$

with $y = (t, x_1, x_2, x_3)$ and $\tilde{A}(y) = \tilde{A} = (\tilde{a}_{jk})_{jk} = \mathrm{diag}(0, -\lambda, -\lambda, -\lambda) \in \mathbb{R}^{4\times 4}$ for all $y = (t,x) \in (0,\infty) \times \Omega$. Since $\tilde{A}$ has the eigenvalue zero with multiplicity 1 and all other eigenvalues have the same sign, we have verified the parabolicity of the heat equation.

Problem 3

a) First, we note that the chain rule implies $u_\varepsilon'(t) = \frac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon(t) = \partial_1 v_\varepsilon(t, \frac{t}{\varepsilon}) + \frac{1}{\varepsilon} \partial_2 v_\varepsilon(t, \frac{t}{\varepsilon})$ and $u_\varepsilon''(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} u_\varepsilon(t) = \partial_1^2 v_\varepsilon(t, \frac{t}{\varepsilon}) + \frac{1}{\varepsilon^2} \partial_2^2 v_\varepsilon(t, \frac{t}{\varepsilon}) + \frac{2}{\varepsilon} \partial_1 \partial_2 v_\varepsilon(t, \frac{t}{\varepsilon})$. Thus, plugging the ansatz into (6) yields

$$\begin{aligned} -\frac{1}{\varepsilon} \lambda'(\tfrac{t}{\varepsilon}) \Big(\partial_1 v_0(t, \tfrac{t}{\varepsilon}) + \tfrac{1}{\varepsilon} \partial_2 v_0(t, \tfrac{t}{\varepsilon}) + \partial_2 v_1(t, \tfrac{t}{\varepsilon}) + \mathcal{O}(\varepsilon) \Big) \\ - \lambda(\tfrac{t}{\varepsilon}) \Big(\frac{2}{\epsilon} \partial_1 \partial_2 v_0(t, \tfrac{t}{\varepsilon}) + \frac{1}{\varepsilon} \partial_2^2 v_1(t, \tfrac{t}{\varepsilon}) + \frac{1}{\varepsilon^2} \partial_2^2 v_0(t, \tfrac{t}{\varepsilon}) + \mathcal{O}(1) \Big) = f(t) \end{aligned}$$

for all $t \in \mathbb{R}$, and for the initial value,

$$\frac{1}{\varepsilon^2} \partial_2^2 v_0(0,0) + \frac{1}{\varepsilon} \left(\partial_2^2 v_1(0,0) + 2\partial_1 \partial_2 v_0(0,0) \right) + \mathcal{O}(1) = \frac{1}{\varepsilon}.$$

Sorting by various orders of ε, we obtain for $\mathcal{O}(\varepsilon^{-2})$,

$$\begin{cases} \lambda'(\tfrac{t}{\varepsilon}) \partial_2 v_0(t, \tfrac{t}{\varepsilon}) + \lambda(\tfrac{t}{\varepsilon}) \partial_2^2 v_0(t, \tfrac{t}{\varepsilon}) = 0, \\ \partial_2^2 v_0(0,0) = 0, \end{cases}$$

and for $\mathcal{O}(\varepsilon^{-1})$,

$$\begin{cases} \lambda'(\tfrac{t}{\varepsilon}) \left(\partial_1 v_0(t, \tfrac{t}{\varepsilon}) + \partial_2 v_1(t, \tfrac{t}{\varepsilon}) \right) + \lambda(\tfrac{t}{\varepsilon}) \left(2\partial_1 \partial_2 v_0(t, \tfrac{t}{\varepsilon}) + \partial_2^2 v_1(t, \tfrac{t}{\varepsilon}) \right) = 0, \\ \partial_2^2 v_1(0,0) + 2\partial_1 \partial_2 v_0(0,0) = 1. \end{cases}$$

b) Let $\mathcal{A} = \{u \in C^2([0,1]) : u''(0) = 1\}$ and define $\mathcal{I} : \mathcal{A} \to \mathbb{R}$ as

$$\mathcal{I}(u) = \int_0^1 \frac{1}{2}\lambda(t)(u'(t))^2 - f(t)u(t) \, \mathrm{d}t, \quad u \in \mathcal{A}.$$

Now, if $\bar{u}$ is a minimizer of $\mathcal{I}$ in $\mathcal{A}$ and $\varphi \in C_c^\infty((0,1))$, then

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathcal{I}(\bar{u} + s\varphi) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_0^1 \frac{1}{2}\lambda(t)(\bar{u}'(t) + s\varphi'(t))^2 - f(t)(\bar{u}(t) + s\varphi(t)) \, \mathrm{d}t \\ &= \int_0^1 \lambda(t)\bar{u}'(t)\varphi'(t) - f(t)\varphi(t) \, \mathrm{d}t \\ &= \int_0^1 \left(-\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\bar{u}'(t)) - f(t) \right) \varphi(t) \, \mathrm{d}t, \end{aligned}$$

where, in the last step, we have used integration by parts. Finally, we conclude from the fundamental lemma in the calculus of variations that

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t)\bar{u}'(t)) = f(t)$$

for every $t \in (0,1)$, which along with $\bar{u}''(0) = 1$ is (6) for $\varepsilon = 1$.

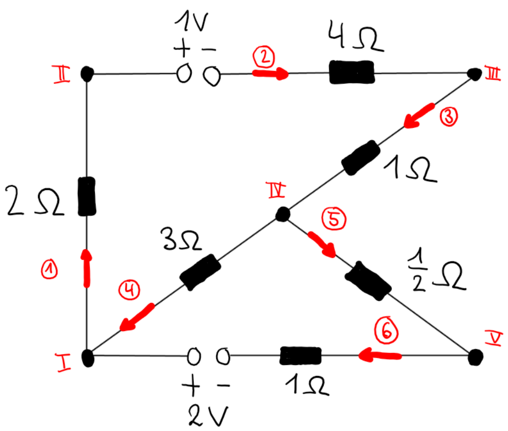
Problem 4

a) Let $y \in \mathcal{A}$ and $t > 0$, then $y + tx \in \mathcal{A}$ if and only if $x \in \mathcal{A}$, since $\mathcal{A} = \ker B^T$ is a vector space. We can thus calculate the first variation to be

$$\delta\mathcal{I}(y)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{I}(y + tx) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (y + tx) \cdot R(y + tx) - b \cdot (y + tx) = (Ry - b) \cdot x,$$

where we have used that R is symmetric. If $y \in \mathcal{A}$ solves (7), then the first variation has to vanish at y, i.e., $\delta\mathcal{I}(y)(x) = 0$ for all $x \in \mathcal{A}$ or equivalently, $(Ry - b) \cdot x = 0$ for all $x \in \ker B^T$.

b) With the following labelling of edges and nodes, and choice of orientation,



we deduce that

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad R = \mathrm{diag}(2, 4, 1, 3, \tfrac{1}{2}, 1), \quad b = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$