Mathematical Modeling

WISB357

Universiteit Utrecht
Blok 2, 2019/20
Dr. Carolin Kreisbeck
Dominik Engl (TA)
Jelle Soons (TA)

Retake Exam

Name:

Student number:

Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van andere hulpmiddelen dan het cursusboek, material uit het werkcollege en eigen aantekeningen.

Signature:

Date: Tuesday, April 14, 2020

Time: 13:30 - 16:30 (3 hours)

Instructions:

- Write your *name, student number, and problem number* on every page you submit electronically.
- Consulting the internet and using calculators, cell phones, etc. is *not* allowed.
- You are permitted to make use of the course book, material from the exercise sessions, and any hand-written notes.
- Make sure that your answers are *readable*, *understandable*, and *well justified.*
- Problems marked with * are bonus questions.

Total points: 60 (including 3 bonus points)

Score:

1	2	3	4	Σ

Grade:

Problem 1

The following system of ordinary differential equations can be used to model the growth of two competing species in an environment of restricted resources:

$$\begin{cases} S_1' &= r_1 S_1 - q_1 S_1^2 - a S_1 S_2, \\ S_2' &= r_2 S_2 - q_2 S_2^2 - b S_1 S_2, \end{cases} \tag{1}$$

where $r_1, r_2 > 0$ are the growth rates of the species $S_1, S_2 \geq 0$, $q_1, q_2 > 0$ reflect the effects of a maximal capacity for the individual species, and $a, b > 0$ correspond to the rates at which the two species eliminate each other.

a) Non-dimensionalize (1) to obtain a system of the form

$$\begin{cases} x_1' &= x_1 - x_1^2 - d_2 x_1 x_2, \\ x_2' &= \rho x_2 - x_2^2 - d_1 x_1 x_2, \end{cases} \tag{2}$$

with $\rho, d_1, d_2 > 0$, and determine these three quantities explicitly in terms of the parameters r_1, r_2, q_1, q_2, a, b. Give an interpretation of ρ in context of the model. 6p

Henceforth, we assume that $d_2 = 1$ and $d_1 = \frac{1}{2}$.

b) Characterize the set of all $\rho > 0$ such that (2) has an asymptotically stable equilibrium that corresponds to a situation where neither of the two species gets extinct. 7p

c) Sketch a phase portrait of (2), including stationary points, isoclines and areas of monotonicity for $\rho = \frac{3}{4}$. 4p

d) Describe and explain the qualitative difference between the long-term behavior of solutions to (2) in the two regimes $\rho = \frac{3}{4}$ and $\rho = 2$. 4p

CONTINUATION ON NEXT PAGE

Problem 2

Consider the boundary-value problem

$$\begin{cases} \partial_t u(t,x) - \nabla \cdot (A(x)\nabla u(t,x)) = f(t,x) & \text{for } (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \partial\Omega, \end{cases} \tag{3}$$

with unknown $u : (0,\infty) \times \Omega \to \mathbb{R}$. Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $f : (0,\infty) \times \Omega \to \mathbb{R}^n$, $A : \Omega \to \mathbb{R}^{3\times 3}$ are smooth functions, and A is supposed to satisfy

$$\xi \cdot A(x)\xi \geq C|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and all } x \in \Omega, \tag{4}$$

with a constant $C > 0$ independent of x and ξ. Note that the differential operators ∇ and $\nabla\cdot$ involve only partial derivatives with respect to the spatial variable $x \in \Omega$.

a) Prove that any solution u to (3) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}|u(t,x)|^2 \, \mathrm{d}x + \int_\Omega \nabla u(t,x) \cdot A(x)\nabla u(t,x) \, \mathrm{d}x = \int_\Omega f(t,x)u(t,x) \, \mathrm{d}x \tag{5}$$

for every $t > 0$. 6p

b) Use (5) and (4) to conclude that, for given initial values, the solution u to (3) is unique.

Hint: Show that the difference of two solutions to (3) is again a solution to (3) for a special choice of f. 6p

c) Specify $A : \Omega \to \mathbb{R}^{3\times 3}$ such that the partial differential equation in (3) turns into the heat equation. 3p

d)* Is the heat equation elliptic, hyperbolic or parabolic? Justify your answer! 3p

CONTINUATION ON NEXT PAGE

Problem 3

Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a smooth, 1-periodic function and let $f : (0,1) \to \mathbb{R}$ be a given smooth function. For $\varepsilon > 0$, we consider the initial-value problem

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\lambda(\tfrac{t}{\varepsilon})u'(t)\big) = f(t), \quad u''(0) = \frac{1}{\varepsilon} \tag{6}$$

for $t \in (0,1)$.

a) Suppose that $u_\varepsilon : (0,1) \to \mathbb{R}$ is a solution to (6) of the form $u_\varepsilon(t) = v_\varepsilon(t, \frac{t}{\varepsilon})$ for all $t \in (0,1)$, where $v_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ can be expressed as

$$v_\varepsilon(x_1,x_2) = v_0(x_1,x_2) + \varepsilon v_1(x_1,x_2) + \varepsilon^2 v_2(x_1,x_2) + \mathcal{O}(\varepsilon^3), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Find the initial value problems for the orders $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ in the asymptotic expansion of u_ε. 6p

b) Determine a set $\mathcal{A}$ and a functional $\mathcal{I}$ such that (6) with $\varepsilon = 1$ is the Euler-Lagrange equation (with boundary values) of the variational problem

$$\text{Minimize } \mathcal{I}(u) \text{ for } u \in \mathcal{A}.$$

6p

Problem 4

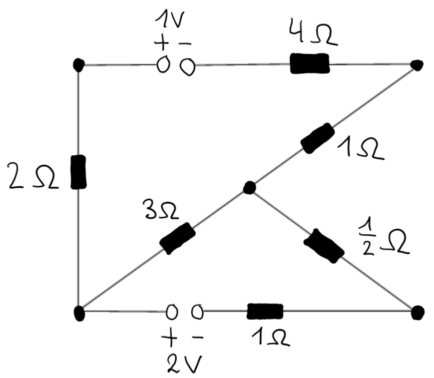
The optimal power within an electrical network can be obtained by solving the optimization problem

$$\text{Minimize } \mathcal{I}(y) = \tfrac{1}{2} y \cdot Ry - b \cdot y \quad \text{for } y \in \mathcal{A} = \{y \in \mathbb{R}^n : B^T y = 0\}, \tag{7}$$

where the matrix of resistances $R \in \mathbb{R}^{n\times n}$ is diagonal and positive definite, $b \in \mathbb{R}^n$ is the voltage source vector, and the incidence matrix $B \in \mathbb{R}^{n\times m}$ describes the network geometry.

a) Interpreting $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ as a functional, identify the (maximal) set of variation directions and compute the first variation of $\mathcal{I}$ at any $y \in \mathbb{R}^n$. Find a necessary condition for solutions to (7). 5p

b) Determine R, B, and b for the following electrical network:



4p