Final exam

Topologie en Meetkunde, Block 3, 2020

Instructions

- You must upload your answers to Blackboard before 16:40. Those with extra time will be allowed an additional 30 minutes.
- You must submit a PDF. You may produce this PDF either using Latex or by scanning a hand-written document. If you do the later, use a high quality scanner.
- Each question has different versions, labeled with letters. Each student has been randomly assigned a version of each exercise; you can find the versions corresponding to you in "Assignments/Final exam". You must answer the version you are given: answering a different version will be evaluated as incorrect.
- The exam is open book: you may use Hatcher or the lecture notes as a reference while working on it. However, you must sign and add to your exam the following declaration: "Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van andere hulpmiddelen dan het cursusboek/dictaat/eigen aantekeningen".
- You have to justify the claims you make. You may use results from the lectures or the book, but you must provide a clear statement (with complete hypothesis and conclusion).
- Pepijn and I are available through Microsoft Teams during the exam. You may send us a private message if you have any questions.
- After the exam is complete, store your physical copy in case it needs to be given to the University for archiving.

1 Exercise 1 (1 point)

Version A. Let $p_1, p_2 \in \mathbb{R}^2$ be distinct points. Denote $A := \mathbb{R}^2 \setminus \{p_1, p_2\}$ and $i_j : A \to \mathbb{R}^2 \setminus \{p_j\}$ the inclusions. Find a non nullhomotopic map $f : \mathbb{S}^1 \to A$ such that $i_j \circ f$ is nullhomotopic for both $j = 1, 2$.

Version B. Find two maps $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \vee_2\mathbb{S}^1$ such that $[\gamma_1] \neq [\gamma_2] \in \pi_1(\vee_2(\mathbb{S}^1, 1))$ but $[\gamma_1] = [\gamma_2] \in [\mathbb{S}^1, \vee_2\mathbb{S}^1]$.

Version C. Find a pointed space (X, q) and an element $[\gamma] \in \pi_1(X, q)$ such that conjugation by $[\gamma]$ is a non-trivial group isomorphism:

$$\beta_{[\gamma]} : \pi_1(X, q) \to \pi_1(X, q).$$

Version D. Let $p \in T^2$. Denote $A := T^2 \setminus \{p\}$ and $i : A \to T^2 \setminus \{p\}$ the inclusion. Find a non nullhomotopic map $f : \mathbb{S}^1 \to A$ such that $i \circ f$ is nullhomotopic.

2 Exercise 2 (1 point)

Provide examples of:

Version A. A space with fundamental group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}^3$.

Version B. A space with fundamental group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Version C. A space with fundamental group $G := \langle g_1, \cdots, g_5 | g_1^2 g_2, g_3 g_4 g_5 \rangle$.

Version D. A space with fundamental group $G := \langle g_1, \cdots, g_4 | g_1^2 g_2^2, g_2 g_3 \rangle$.

3 Exercise 3 (1 point)

Provide examples of:

Version A. A pair of non-homeomorphic surfaces which are homotopy equivalent.

Version B. A pair of non-homeomorphic compact surfaces with the same Euler characteristic.

Version C. A pair of non-homeomorphic path-connected compact surfaces A and B such that A is a covering space of B.

Version D. A compact, connected, non-orientable surface with Euler characteristic -4.

Version E. A compact, connected, orientable surface with Euler characteristic -6.

4 Exercise 4 (1 point)

Version A. Are there non-homeomorphic compact surfaces A and B such that $A \# T^2$ and $B \# T^2$ are homeomorphic?

Version B. Let M denote the Möbius band. Are there non-homeomorphic compact surfaces A and B such that $A \# M$ and $B \# M$ are homeomorphic?

Version C. Let C denote the cylinder. Are there non-homeomorphic compact surfaces A and B such that $A \# C$ and $B \# C$ are homotopy equivalent?

5 Exercise 5 (2 points)

Let S be the surface with planar representation as in the Figure below (the versions are indicated there). Determine all the g and g' such that S is homeomorphic to Σ_g or $N_{g'}$. What is its Euler characteristic? Is it orientable?

6 Exercise 6 (2 points)

Consider the space X below. Then:

- Endow it with a CW-complex structure.
- Compute its Euler characteristic.
- Compute its universal cover $\tilde{X}$.
- Describe the CW-structure that $\tilde{X}$ inherits from X through the covering map. State explicitly how many cells it has and how they are attached to one another.
- Draw (schematically) both CW-complexes, labelling the different cells.

Version A. Let $X := (T^2, p) \vee (\mathbb{D}^2, q)$. Here q is a point in the interior of $\mathbb{D}^2$.

Version B. Let $X := (\mathbb{RP}^2, q) \vee (\mathbb{RP}^2, q)$.

Version C. Let $X := (\mathbb{RP}^3, p) \vee (\mathbb{D}^2, q) \vee (\mathbb{S}^2, n)$. Here q is a point in the interior of $\mathbb{D}^2$.

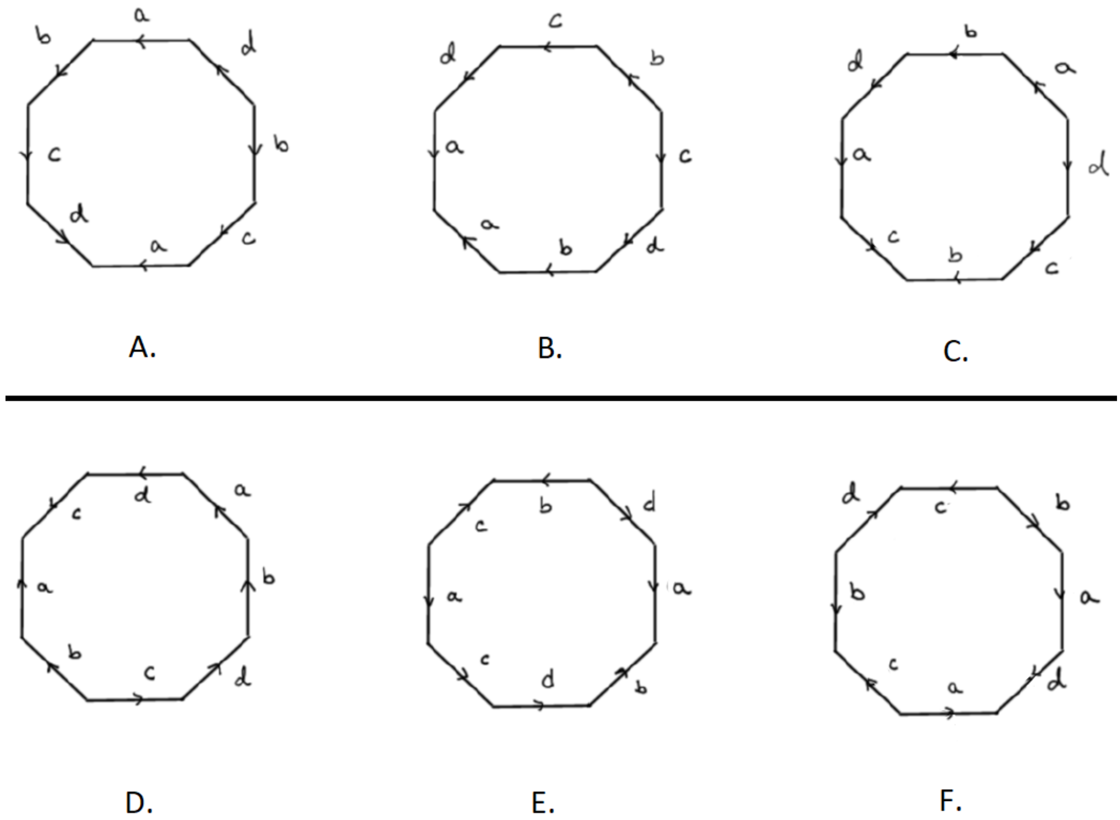

Figure 1: Planar representation for Exercise 5.

7 Exercise 7 (2 points)

For each (possibly disconnected) pointed covering space in the versions below:

- Compute the subgroup of the fundamental group of the base it corresponds to.
- Describe its deck transformations.
- Determine whether it is a normal covering space.

Version A. Enumerate, up to isomorphism, all the 2-sheeted pointed covering spaces of $\vee_m(\mathbb{S}^1, 1)$, m an integer.

Version B. Enumerate, up to isomorphism, all the 2-sheeted pointed covering spaces of $\vee_m(\mathbb{RP}^2, q)$, m an integer.

Version C. Enumerate, up to isomorphism, all the 3-sheeted pointed covering spaces of $\vee_2(\mathbb{S}^1, 1)$.

Version D. Enumerate, up to isomorphism, all the 3-sheeted pointed covering spaces of $(\mathbb{S}^1, 1) \vee (\mathbb{RP}^2, q)$.