Retake

Topologie en Meetkunde, Block 3, 2020

Instructions

- You have two hours to prepare answers to the questions below. Those with extra time will be allotted an additional 15 minutes.
- You will have an hour to defend your answers in an oral examination through Teams. Those with extra time will be allotted an additional 15 minutes.
- You do not have to submit the written notes you have produced during the preparation.
- The exam is **not** open book. For monitoring purposes, we will be connected through Teams during the preparation (with the camera on). Do not interact with your computer for the duration of the exam.
- We will be available during the exam (myself first and later Pepijn) to answer questions. If there are other students working, and in order not to disturb them, send us a private message (i.e. do not ask through the video chat).
- You should have a suitable way of presenting your work to us. In particular, you should be able to write in a manner that is not distracting (use a tablet or use your phone to record your paper, projector-style).
- When you present, be structured. Write and state clearly what steps you take and what results (from the lectures or the book) you use.

Questions

Exercise 1 (5.5 points)**.** Prove, or provide a counterexample to, the following statements:

A. A space A is contractible if and only if $[A, A] = \{.\}$.

B. Let A and B be homotopy equivalent spaces. Then A is compact if and only if B is compact.

C. Let $A \subset \mathbb{R}$ be a discrete collection of points with cardinality $|A| > 1$. Then A is not a retract of $\mathbb{R}$.

D. There are subsets $A \subset \mathbb{R}$ which are not deformation retracts of any neighbourhood $B \supset A$.

E. Let A and B be spaces. If $f : A \to B$ induces isomorphisms

$$f_* : \pi_1(A, p) \quad \to \quad \pi_1(B, f(p))$$

for all $p \in A$, then

$$f_* : \Pi_1(A) \quad \to \quad \Pi_1(B)$$

is a bijection.

F. There are finitely many distinct triangulations of $\mathbb{S}^1$, up to simplex-preserving homeomorphism.

G. A non-orientable surface cannot be simply-connected.

H. Every map $\mathbb{S}^2 \to \mathbb{S}^1$ is contractible.

I. All the covering spaces of the Klein bottle are not orientable.

J. Let X and Y be path-connected, locally contractible topological spaces. The universal cover of $X \times Y$ is the product of the universal covers of each factor.

K. Let $K \subset Y$ be a deformation retract. Let $p : \tilde{X} \to X$ be a covering space and let $f : Y \to X$ be a map. Assume all spaces involved are path-connected and locally contractible. Show that $g = f|_K : K \to X$ lifts to a map $\tilde{g} : K \to \tilde{X}$ if and only if f lifts to $\tilde{f} : Y \to \tilde{X}$.

Exercise 2 (2 points)**.** Let A, B be two copies of the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Given the space

$$C := (A \coprod B)/(A \ni (z, 1) \cong (z^2, z^3) \in B),$$

- Compute its fundamental group.
- Endow it with a CW-structure (it is best if you show this pictorially). State explicitly how many cells of each dimension you use.

Exercise 3 (2.5 points)**.** Consider the space $X_{p,q} := (\mathbb{RP}^2, p) \vee (\mathbb{S}^2, q)$.

- Show that, for different choices of points p and q, the resulting spaces $X_{p,q}$ are all homeomorphic.
- Compute its fundamental group.
- Describe all its connected covering spaces (up to pointed isomorphism). Compute their fundamental groups and deck transformations.
- Let $Y_{a,b} := X_{p,q} \setminus \{a, b\}$, where $a \neq b$ are points different from the wedge point. What is the fundamental group of $Y_{a,b}$? Does the result depend on the choice of a and b?