Midterm exam

Topologie en Meetkunde, Block 3, 2020

Instructions

- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (1,5 points)**.** Let $C := [-1,1]^2 \subset \mathbb{R}^2$ be the square and let

$$\partial C := ([-1,1] \times \{-1,1\}) \cup (\{-1,1\} \times [-1,1])$$

be its boundary. Write a explicit deformation retraction of $C \setminus \{(0,0)\}$ to ∂C.

Exercise 2 (1,5 points)**.** Let $B := \mathbb{S}^1 \cup (\{0\} \times [0,2]) \subset \mathbb{R}^2$. Show that B is homotopy equivalent to $\mathbb{S}^1$ but not homeomorphic to it.

Exercise 3 (1,5 points)**.** Let X, Y and W be topological spaces. Let $f : X \to Y$ be a continuous map. Recall that $[Y,W]$ denotes the set of equivalence classes of continuous maps $Y \to W$ up to homotopy. We define the **pullback** of f to be:

$$\begin{aligned} f^* : [Y,W] &\longrightarrow [X,W], \\ [g] &\longrightarrow f^*([g]) := [g \circ f]. \end{aligned}$$

Show that:

- f^* is a well-defined function.
- Given homotopic maps $f_0, f_1 : X \to Y$, it follows that $f_0^* = f_1^*$.
- If f is a homotopy equivalence then f^* is a bijection.

Exercise 4 (1,5 points)**.** Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the sphere. Let $\mathbb{S}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$ be its equator. Show that there is no retraction $r : \mathbb{S}^2 \to \mathbb{S}^1$.

Exercise 5 (1 points)**.** Let $C := [-1,1]^2 \subset \mathbb{R}^2$ be the square. Show that the following paths are homotopic relative to their endpoints:

$$\begin{aligned} \gamma_0, \gamma_1 : [0,1] &\to C \\ \gamma_0(s) &:= (2s-1,-1), \\ \gamma_1 &:= (s \to (1, 1-2s)) \bullet (s \to (2s-1, 1)) \bullet (s \to (-1, 2s-1)). \end{aligned}$$

Construct a explicit homotopy. Suggestion: draw the paths involved.

Exercise 6 (1,5 points)**.** Let A and B be two copies of $\mathbb{R}$. Let k be a positive integer. The line with k double points is

$$L_k := \left(A \coprod B\right) / \Big((A \setminus \{1, \cdots, k\}) \ni x \cong x \in (B \setminus \{1, \cdots, k\})\Big).$$

The fundamental group of L_1 is $\pi_1(L, p) \cong \mathbb{Z}$ for all $p \in L_1$. Using this information (which you do not have to prove), show that the fundamental group of L_k is isomorphic to $*_k\mathbb{Z}$ (i.e. the group with k generators and no relations).

Exercise 7 (1,5 points)**.** Let $A := \mathbb{S}^2 \cup (\{0\} \times \{0\} \times [-1, 1]) \subset \mathbb{R}^3$. Compute the fundamental group of A.