Retake Exam
Representation of Finite Groups
13.07.2020, 9:00-12:00

Important

You have to upload your solution until 12:30.
Students with extra time have 30 minutes more. If that is true for you, you have to upload until 13:00.

You have to add the following declaration to your solution:

Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van het internet.

There are 43 points to earn in the exam. 3 of those are bonus points. That means you receive the maximal grade if you get 40 or more points. You can use the statement of previous parts of a question even if you were not able to prove them.

Question 1

Let $G = \langle a, b, c : a^3 = b^2 = c^2 = 1, bc = cb, cac = bab = a^2 \rangle$. It can be shown that this group has 12 elements of the form $a^k b^i c^j$ with $k \in \{0, 1, 2\}$, and $i, j \in \{0, 1\}$.

a) Determine the 6 conjugacy classes of G. (3 points)

b) Find all linear characters of G. (2 points)

c) Show that there exists a representation of G, say $\rho : G \to GL(\mathbb{C}, 2)$ such that

$$\rho(a) = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{pmatrix} \qquad \rho(b) = \rho(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(2 points)

d) Calculate the character of ρ and show that it is irreducible. (2 points)

e) Complete the character table of G. (3 points)

Question 2

Let G be a finite group and $G' = [G, G]$ its commutator subgroup. We set $G_{ab} = G/G'$.

a) Let $g \in G$ but $g \notin G'$. Show that there exists a linear character χ of G such that $\chi(g) \neq 1$. (Hint: Use character orthogonality for characters of G_{ab} and the results of chapter 17.) (4 points)

b) Let $n > 1$ and assume that G only has one irreducible character of dimensions n, say ψ. Show that $\psi(g) = 0$ for all $g \notin G'$. (Hint: Use a) and multiply characters.) (3 points)

Question 3

Let $G = S_4$ be the group of permutations of four objects. Let further $V = \mathbb{C}[X_1, X_2, X_3, X_4]$ be the (infinite dimensional) complex vector space of polynomials in four variables. We describe an action of G on V as follows. For $p(x_1, x_2, x_3, x_4) \in V$ and $\sigma \in G$ we define

$$\sigma p(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}).$$

We consider the following finite dimensional subspaces of V:

$$A = \{ \sum_{1 \le i \le 4} \lambda_i x_i : \lambda_i \in \mathbb{C}\}$$
$$B = \{ \sum_{1 \le i \le j \le 4} \lambda_{i,j} x_i x_j : \lambda_{i,j} \in \mathbb{C}\}$$

With the inherited action from V they become $\mathbb{C}G$-modules. You can find the character table for S_4 in 18.1 in the book.

a) Determine the characters of A and B. (2 points)

b) Dissect the character of A into irreducible characters. (1 point)

c) Show that B contains a submodule isomorphic to A in two ways: By giving an explicit isomorphism and by using characters. (3 points)

d) Give a basis for the submodules of B that are isomorphic to the trivial module. (Hint: Use 14.26 or skillful guessing.) (3 points)

Consider now $C = A \otimes A$. This becomes a $\mathbb{C}G$-module as described in chapter 19.

e) Dissect the character of C into irreducible characters. (3 points)

f) Find a basis for all submodules of C that are isomorphic to the trivial module. (Hint: Compare to d)) (2 points)

Question 4

Let G be a finite group and let χ be any character of G. We set $z = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})g$, where we take the sum in the group algebra $\mathbb{C}[G]$. Let U be an *irreducible* $\mathbb{C}G$-module an consider the linear map

$$\xi : U \to U$$
$$u \mapsto zu.$$

a) Show that $hzh^{-1} = z$ for any $h \in G$. (2 points)

b) Show that there exists a $\lambda \in \mathbb{C}$ such that $\xi(u) = \lambda u$. (Hint: Show that ξ is a $\mathbb{C}G$-homomorphism) (2 points)

c) Let ψ be the character of U. Show that $\frac{\langle \psi, \chi \rangle}{\lambda} = \psi(1)$. (Hint: Calculate the trace of ξ in two ways.) (3 points)

d) Let now χ be irreducible. Show that considered in the group algebra it holds that $z^2 = \frac{z}{\chi(1)}$. (3 bonus points)