Final Exam – Elementaire Getaltheorie

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Problem 1 (9 points). *Decide for* 127 *and* $2^{12} + 1 = 4097$ *whether they are prime.*

Problem 2 (5 points). *Give an example of a primitive root modulo* 7*.*

Problem 3 (24 points).

(a) *Show that* $\sqrt{11}$ *is irrational. (Without citing a theorem from the lecture on October 31.)*

(b) *Show that for any* $p, q \in \mathbb{N}$*, we have* $|\sqrt{11} - \frac{p}{q}| > \frac{1}{8q^2}$
(One-point bonus variant: prove it with 7 *instead of* 8*)*

(c) *Give a fraction* $\frac{p}{q}$ *with* $p, q \in \mathbb{N}$ *such that* $0 < |\sqrt{11} - \frac{p}{q}| < \frac{1}{3600}$*.*

Problem 4 (8 points). *Give two pairs* (x, y) *of positive integers such that* $11y^2 = x^2 + 2x$*.*

Problem 5 (15 points). *Decide for the following two equations whether they have infinitely many solutions* (x, y) *with* $x, y \in \mathbb{Q}$*:*

(a) $x^2 + y^2 = 245$

(b) $y^4 = x^4 + 1$

Problem 6 (8 points). *Show that* $y^2 = 29x^2 + 11$ *does not have solutions with* $x, y \in \mathbb{Z}$*.*

Problem 7 (12 points, of which 6 are bonus). *Show that there is an integer* n *between* 219 *and* 2019 *such that* n *divides* $2^n + 2$*. (Hint:* n *can be chosen to be of the form* $2pq$ *with* p *and* q *primes.)*[1]

No calculator is allowed. All statements must be supported by arguments or citation from class.

[1] This problem is a difficult problem, mostly meant to distinguish a grade of 9 from a grade of 10.