

Solutions for first midterm

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In the solutions, I do not give all possible approaches, but just one or two. Please write to me if you find any typos or have other remarks.

Problem 1 (4 points). Write $\frac{1153}{140}$ as a continued fraction.

We need to do the Euclidean algorithm:

$$1153 = 8 \cdot 140 + 33$$

$$140 = 4 \cdot 33 + 8$$

$$33 = 4 \cdot 8 + 1$$

$$8 = 8 \cdot 8 + 0.$$

Thus, the continued fraction expansion is $[8, 4, 4, 8]$.

Problem 2 (8 points). Determine for $n = 1236$ and $n = 1153$ whether they are the sum of two squares. If yes, find one pair of integers (x, y) with $n = x^2 + y^2$. (Hint: You may use that $140^2 + 1$ is divisible by 1153.)

We know that each number is congruent to its digit sum modulo 9. Thus,

$$1236 \equiv 12 \equiv 3 \pmod{9}.$$

In particular, we see that the exponent of 3 in the prime factorization of 1236 is 1, thus odd. Thus, 1236 is not the sum of two squares. [Alternative: The squares modulo 9 are 0, 1, 4 and 7. No sum of two of these is 3.]

In contrast, 1153 is $33^2 + 8^2 = 1089 + 64$. We used Cornacchia's algorithm to find these numbers.

Problem 3 (8 points). Show that for every natural number $a > 2$, there are natural numbers b, c with $a^2 + b^2 = c^2$. (Hint: Distinguish the cases a even and a odd.)

We first note that $(s^2 - t^2)^2 + (2st)^2 = (s^2 + t^2)^2$.

Let a be odd. We can write it as $2s + 1$ with $s \in \mathbb{N}$. Setting $t = s + 1$, we observe that $a = s^2 - t^2$. Set $b = 2st$ and $c = s^2 + t^2$.

Let a be even. We can write it as $2s$ with $s \in \mathbb{N}$. Setting $t = 1$, we observe that $a = 2st$. Set $b = s^2 - t^2$ and $c = s^2 + t^2$.

Problem 4 (8 points). *Let n be a positive integer of the form $8k + 7$ with $k \in \mathbb{Z}$. Show that n is of the form $x^2 + y^2 + z^2 + w^2$ with x, y, z and w **positive** integers.*

First of all we know by Lagrange's four-square theorem that we can find (not necessarily positive) integers x, y, z and w with $n = x^2 + y^2 + z^2 + w^2$. As $(-x)^2 = x^2$, we can assume that $x, y, z, w \geq 0$.

Let $a \in \mathbb{Z}$. We have seen in class that a^2 is congruent to 0, 1 or 4 modulo 8. This implies that x^2, y^2, z^2 and w^2 must be (up to reordering) congruent to 1, 1, 1 and 4 modulo 8. Thus, x, y, z and w are nonzero and thus positive.