Measure and Integration: Solutions Final 2019-20

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra and λ is Lebesgue measure. For $n \geq 1$, let $u_n(x) = \mathbb{I}_{[0,1-2^{-n})}(x) \cos(e^{-x/n})\, x^2$.

(a) Prove that $\lim_{n\to\infty} \int u_n \, d\lambda = \frac{1}{3}$. (1 pt)

(b) Let $1 < p < \infty$, prove that $\left|\sum_{n=1}^{\infty} \left(\frac{u_n}{n}\right)^p\right| < \infty$ μ a.e. (1 pt)

Proof (a) Let $u(x) = \mathbb{I}_{[0,1)} x^2$, since u is Riemann integrable, then $u \in \mathcal{L}^1(\lambda)$ and

$$\int u \, d\lambda = (R)\int_0^1 x^2 \, dx = \frac{1}{3}.$$

We have $\lim_{n\to\infty} u_n(x) = \mathbb{I}_{[0,1)}(x) x^2 \cos(0) = \mathbb{I}_{[0,1)}(x) x^2 = u(x)$ and $|u_n(x)| \leq u(x)$. Thus, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int u_n \, d\lambda = \int \lim_{n\to\infty} u_n \, d\lambda = \int u \, d\lambda = \frac{1}{3}.$$

Proof (b) For $1 < p < \infty$, we have

$$\begin{aligned}
\int |u_n|^p \, d\lambda &= \int \mathbb{I}_{[0,1-2^{-n})}(x) |\cos(e^{-x/n})|^p \, x^{2p} \, d\lambda(x) \\
&\leq \int \mathbb{I}_{[0,1)}(x)\, x^{2p} \, d\lambda(x) \\
&= (R)\int_0^1 x^{2p} \, dx \\
&= \frac{1}{2p+1}.
\end{aligned}$$

Thus, by Corollary 9.9 and the fact that $1 < p < \infty$ we have

$$\begin{aligned}
\int \sum_{n=1}^{\infty} \left(\frac{|u_n|}{n}\right)^p d\lambda &= \sum_{n=1}^{\infty} \int \frac{|u_n|^p}{n^p} d\lambda \\
&\leq \frac{1}{2p+1} \sum_{n=1}^{\infty} \frac{1}{n^p} \\
&< \infty.
\end{aligned}$$

By Corollary 11.6, $\sum_{n=1}^{\infty} \left(\frac{|u_n|}{n}\right)^p < \infty$ μ a.e. Since

$$\left|\sum_{n=1}^{\infty} \left(\frac{u_n}{n}\right)^p\right| \leq \sum_{n=1}^{\infty} \left(\frac{|u_n|}{n}\right)^p,$$

it follows that $\left|\sum_{n=1}^{\infty} \left(\frac{u_n}{n}\right)^p\right| < \infty$ μ a.e.

(2) Let $(X, \mathcal{A}, \mu)$ be a **finite** measure space, and $1 < p, q < \infty$ two conjugate numbers (i.e. $1/p + 1/q = 1$). Let $v \in \mathcal{M}(\mathcal{A})$ be a measurable function satisfying

$$\int |uv|\, d\mu \leq ||u||_q$$

for all $u \in \mathcal{L}^q(\mu)$.

(a) For $n \geq 1$, let $A_n = \{x \in X : |v(x)| \leq n\}$ and $v_n = \mathbb{I}_{A_n} |v|^{p/q}$. Prove that $v_n \in \mathcal{L}^q(\mu)$ and that $||v_n||_q^q = ||\mathbb{I}_{A_n} v||_p^p$ for all $n \geq 1$. (0.75 pts)

(b) Prove that $||\mathbb{I}_{A_n} v||_p \leq 1$ for all $n \geq 1$. (1.5 pt)

(c) Prove that $v \in \mathcal{L}^p(\mu)$. (0.75 pt)

Proof (a): Since $|v_n|^q = \mathbb{I}_{A_n} |v|^p$, we have $||v_n||_q^q = ||\mathbb{I}_{A_n} v||_p^p$ and

$$\int |v_n|^q\, d\mu = \int \mathbb{I}_{A_n} |v|^p\, d\mu \leq n^p \mu(A_n) < \infty,$$

for all $n \geq 1$.Thus, $v_n \in \mathcal{L}^q(\mu)$ for all $n \geq 1$.

Proof (b): Notice that $|v_n v| = \mathbb{I}_{A_n} |v|^{\frac{p}{q}+1} = \mathbb{I}_{A_n} |v|^p$ for $n \geq 1$. Since $v_n \in \mathcal{L}^q(\mu)$, then by hypothesis and part (a) we have

$$||\mathbb{I}_{A_n} v||_p^p = \int |v_n v|\, d\mu \leq ||v_n||_q = ||\mathbb{I}_{A_n} v||_p^{p/q}.$$

Dividing both sides by $||\mathbb{I}_{A_n} v||_p^{p/q}$ and using the fact that p and q are conjugates we obtain $||\mathbb{I}_{A_n} v||_p \leq 1$.

Proof (c): Since (A_n) is an exhausting sequence of measurable sets with $\bigcup_{n=1}^{\infty} A_n = X$, then $\mathbb{I}_{A_n} |v|^p \nearrow |v|^p$. By Beppo-Levi we have $||v||_p \leq 1$, and hence $v \in \mathcal{L}^p(\mu)$.

(3) Consider the product space $([1,2], \mathcal{B}([1,2]) \otimes \mathcal{B}((0,\infty) \lambda \otimes \lambda)$ with λ is Lebesgue measure restricted on the appropriate spaces. Consider the fuction $f : [1,2] \times (0,\infty) \to (0,\infty)$ defined by $f(x,t) = e^{-xt}$.

(a) Prove that $f \in \mathcal{L}^1(\lambda \otimes \lambda)$. (1pt)

(b) Prove that $\int_{(0,\infty)} (e^{-t} - e^{-2t}) \frac{1}{t} d\lambda(t) = \log(2)$. (1pt)

Proof (a) Let $f : [1,2] \times (0,\infty)$ be given by $f(x,t) = e^{-xt}$. Then f is continuous (hence measurable) and $f > 0$. For each fixed $x \in [1,2]$, the function $t \to e^{-xt}$ is positive measurable and the improper Riemann integrable on $[0,\infty)$ exists, so that

$$\int_{(0,\infty)} e^{-xt} d\lambda(t) = (R) \int_0^\infty e^{-xt} dt = \frac{1}{x}.$$

Furthermore, the function $x \to \frac{1}{x}$ is measurable and Riemann integrable on $[1,2]$, thus

$$\int_{[1,2]} \int_{(0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x) = \int_{[1,2]} \frac{1}{x}\, d\lambda(x) = (R) \int_1^2 \frac{1}{x}\, dx = \log(2) < \infty.$$

Thus, by Fubini's Theorem $f \in \mathcal{L}^1(\lambda \otimes \lambda)$ and $\int_{[1,2] \times (0,\infty)} f\, d(\lambda \times \lambda) = \log(2)$.

Proof (b) By Toneli's Theorem (or Fubini),

$$\int_{(0,\infty)} \int_{[1,2]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[1,2]} \int_{(0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x).$$

We have,

$$\int_{(0,\infty)} \int_{[1,2]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{(0,\infty)} \int_{[1,2]} e^{-xt} dx\, d\lambda(t) = \int_{(0,\infty)} (e^{-t} - e^{-2t}) \frac{1}{t} d\lambda(t).$$

Therefore, $\int_{(0,\infty)} (e^{-t} - e^{-2t})\frac{1}{t} d\lambda(t) = \log(2)$.

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{M}(\mathcal{A})$ satisfies $u^n \in \mathcal{L}^1(\mu)$ for all $n \geq 1$.

(a) Suppose $\lim_{n\to\infty} \int u^n \, d\mu$ exists and is finite. For $k \geq 1$, let $E_k = \{x \in X : |u(x)| > 1 + 1/k\}$. Prove that $\int u^{2n} \, d\mu \geq (1 + 1/k)^{2n} \mu(E_k)$ for all $n \geq 1$. (0.75 pts)

(b) Let $E = \{x \in X : |u(x)| > 1\}$. Prove that $\mu(E) = 0$ and conclude that $|u(x)| \leq 1$ μ a.e. (Hint: give a proof by contradiction) (1.25 pts)

(c) Prove that $\int u^n \, d\mu = c$ is a constant for all $n \geq 1$ **if and only if** $u = \mathbb{I}_A$ μ a.e. for some measurable set $A \in \mathcal{A}$. (Hint: consider the function $u^2(1-u)^2$) (1 pt)

Proof (a) For any $n \geq 1$,

$$u^{2n} = u^{2n}\mathbb{I}_{E_k} + u^{2n}\mathbb{I}_{E_k^c} \geq u^{2n}\mathbb{I}_{E_k} \geq (1 + 1/k)^{2n}\mathbb{I}_{E_k}.$$

Thus, for all $n \geq 1$

$$\int u^{2n} \, d\mu \geq (1 + 1/k)^{2n} \mu(E_k).$$

Proof (b) Assume for the sake of getting a contradiction that $\mu(E) > 0$. Note the sequence $(E_k)_{k\in\mathbb{N}}$ as given in part (a), is an increasing sequence of measurable sets with $E = \bigcup_{k=1}^{\infty} E_k$. Since $\mu(E) > 0$, then there exists $k \geq 1$ sufficiently large such that $\mu(E_k) > 0$. From part (a), we have for all $n \geq 1$

$$\int u^{2n} \, d\mu \geq (1 + 1/k)^{2n} \mu(E_k).$$

This implies that

$$\lim_{n\to\infty} \int u^{2n} \, d\mu \geq \lim_{n\to\infty} (1 + 1/k)^{2n} \mu(E_k) = \infty,$$

contradicting the fact that $\lim_{n\to\infty} \int u^n \, d\mu < \infty$. Thus $\mu(E) = 0$ and $|u(x)| \leq 1$ μ a.e.

Proof (c) If $u = \mathbb{I}_A$ for some measurable set $A \in \mathcal{A}$, then $u^n = \mathbb{I}_A$ for all $n \geq 1$ and hence $\int u^n \, d\mu = \mu(A) < \infty$ for all $n \geq 1$. Therefore, the result holds with $c = \mu(A)$.

Conversely, assume $\int u^n \, d\mu = c$ is a constant for all $n \geq 1$. Note that

$$\int u^2(1-u)^2 \, d\mu = \int u^2 \, d\mu - 2\int u^3 \, d\mu + \int u^4 \, d\mu = c - 2c + c = 0.$$

Since $u^2(1-u)^2 \geq 0$, this implies that $u^2(1-u)^2 = 0$ μ a.e. implying that u takes only two values 0 and 1 μ a.e. Equivalently, u is μ a.e.equals the indicator function $\mathbb{I}_A$ with $A = \{x \in X : u(x) = 1\}$.