Measure and Integration: Solutions Final 2019-20

(1) Consider the measure space \((\mathbb{R}, \mathcal{B}, \lambda)\), where \(\mathcal{B}\) is the Borel \(\sigma\)-algebra and \(\lambda\) is Lebesgue measure. For \(n \geq 1\), let \(u_n(x) = \mathbb{I}_{[0, 1 - 2^{-n})}(x) \cos(e^{-x/n}) x^2\).

(a) Prove that \(\lim_{n \to \infty} \int u_n d\lambda = \frac{1}{3}\). (1 pt)

(b) Let \(1 < p < \infty\), prove that \(\left| \sum_{n=1}^{\infty} \left( \frac{|u_n|}{n} \right)^p \right| < \infty\) a.e. (1 pt)

Proof (a) Let \(u(x) = \mathbb{I}_{[0,1)} x^2\), since \(u\) is Riemann integrable, then \(u \in L^1(\lambda)\) and
\[
\int u \, d\lambda = (R) \int_0^1 x^2 \, dx = \frac{1}{3}.
\]
We have \(\lim_{n \to \infty} u_n(x) = \mathbb{I}_{[0,1)}(x)x^2 \cos(0) = \mathbb{I}_{[0,1)}(x)x^2 = u(x)\) and \(|u_n(x)| \leq u(x)\). Thus, by Lebesgue Dominated Convergence Theorem,
\[
\lim_{n \to \infty} \int u_n \, d\lambda = \int \lim_{n \to \infty} u_n \, d\lambda = \int u \, d\lambda = \frac{1}{3}.
\]

Proof (b) For \(1 < p < \infty\), we have
\[
\int |u_n|^p \, d\lambda = \int \mathbb{I}_{[0,1-2^{-n})}(x) | \cos(e^{-x/n})|^p x^2 \, d\lambda(x)
\leq \int \mathbb{I}_{[0,1)}(x) x^{2p} \, d\lambda(x)
= (R) \int_0^1 x^{2p} \, dx
= \frac{1}{2p+1}.
\]
Thus, by Corollary 9.9 and the fact that \(1 < p < \infty\) we have
\[
\int \sum_{n=1}^{\infty} \left( \frac{|u_n|}{n} \right)^p \, d\lambda = \sum_{n=1}^{\infty} \int \frac{|u_n|^p}{n^p} \, d\lambda
\leq \frac{1}{2p+1} \sum_{n=1}^{\infty} \frac{1}{n^p}
< \infty.
\]
By Corollary 11.6, \(\sum_{n=1}^{\infty} \left( \frac{|u_n|}{n} \right)^p < \infty\) a.e. Since
\[
\left| \sum_{n=1}^{\infty} \left( \frac{u_n}{n} \right)^p \right| \leq \sum_{n=1}^{\infty} \left( \frac{|u_n|}{n} \right)^p,
\]
it follows that \(\sum_{n=1}^{\infty} \left( \frac{u_n}{n} \right)^p < \infty\) a.e.
(2) Let \((X, A, \mu)\) be a finite measure space, and \(1 < p, q < \infty\) two conjugate numbers (i.e. \(1/p + 1/q = 1\)). Let \(v \in \mathcal{M}(A)\) be a measurable function satisfying

\[
\int |uv| \, d\mu \leq ||u||_q
\]

for all \(u \in L^q(\mu)\).

(a) For \(n \geq 1\), let \(A_n = \{x \in X : |v(x)| \leq n\}\) and \(v_n = \mathbb{I}_{A_n} |v|^{p/q}\). Prove that \(v_n \in L^q(\mu)\) and

\[
||v_n||_q^q = ||\mathbb{I}_{A_n} v||_p^p
\]

for all \(n \geq 1\). (0.75 pts)

(b) Prove that \(||\mathbb{I}_{A_n} v||_p \leq 1\) for all \(n \geq 1\). (1.5 pt)

(c) Prove that \(v \in L^p(\mu)\). (0.75 pt)

**Proof (a):** Since \(|v_n|^q = \mathbb{I}_{A_n} |v|^p\), we have \(||v_n||_q^q = ||\mathbb{I}_{A_n} v||_p^p\) and

\[
\int |v_n|^q \, d\mu = \int \mathbb{I}_{A_n} |v|^p \, d\mu \leq n^p \mu(A_n) < \infty,
\]

for all \(n \geq 1\). Thus, \(v_n \in L^q(\mu)\) for all \(n \geq 1\).

**Proof (b):** Notice that \(|v_n v| = \mathbb{I}_{A_n} |v|^{p/q+1} = \mathbb{I}_{A_n} |v|^p\) for \(n \geq 1\). Since \(v_n \in L^q(\mu)\), then by hypothesis and part (a) we have

\[
||\mathbb{I}_{A_n} v||_p^p = \int |v_n v| \, d\mu \leq ||v_n||_q = ||\mathbb{I}_{A_n} v||_p^{p/q}.
\]

Dividing both sides by \(||\mathbb{I}_{A_n} v||_p^{p/q}\) and using the fact that \(p\) and \(q\) are conjugates we obtain \(||\mathbb{I}_{A_n} v||_p \leq 1\).

**Proof (c):** Since \((A_n)\) is an exhausting sequence of measurable sets with \(\bigcup_{n=1}^{\infty} A_n = X\), then \(\mathbb{I}_{A_n} |v|^p \nearrow |v|^p\). By Beppo-Levi we have \(||v||_p \leq 1\), and hence \(v \in L^p(\mu)\).

(3) Consider the product space \(([1, 2], B([1, 2]) \otimes B((0, \infty) \lambda \otimes \lambda))\) with \(\lambda\) is Lebesgue measure restricted on the appropriate spaces. Consider the function \(f : [1, 2] \times (0, \infty) \to (0, \infty)\) defined by \(f(x, t) = e^{-xt}\).

(a) Prove that \(f \in L^1(\lambda \otimes \lambda)\). (1pt)

(b) Prove that \(\int_{(0, \infty)} (e^{-t} - e^{-2t}) \frac{1}{t} \, d\lambda(t) = \log(2)\). (1pt)

**Proof (a):** Let \(f : [1, 2] \times (0, \infty)\) be given by \(f(x, t) = e^{-xt}\). Then \(f\) is continuous (hence measurable) and \(f > 0\). For each fixed \(x \in [1, 2]\), the function \(t \to e^{-xt}\) is positive measurable and the improper Riemann integrable on \([0, \infty)\) exists, so that

\[
\int_{(0, \infty)} e^{-xt} \, d\lambda(t) = (R) \int_0^\infty e^{-xt} \, dt = \frac{1}{x}.
\]

Furthermore, the function \(x \to \frac{1}{x}\) is measurable and Riemann integrable on \([1, 2]\), thus

\[
\int_{[1, 2]} \int_{(0, \infty)} e^{-xt} \, d\lambda(t) \, d\lambda(x) = \int_{[1, 2]} \frac{1}{x} \, d\lambda(x) = (R) \int_1^2 \frac{1}{x} \, dx = \log(2) < \infty.
\]

Thus, by Fubini’s Theorem \(f \in L^1(\lambda \otimes \lambda)\) and \(\int_{[1, 2] \times (0, \infty)} f \, d(\lambda \times \lambda) = \log(2)\).

**Proof (b):** By Tonelli’s Theorem (or Fubini),

\[
\int_{(0, \infty)} \int_{[1, 2]} e^{-xt} \, d\lambda(x) \, d\lambda(t) = \int_{[1, 2]} \int_{(0, \infty)} e^{-xt} \, d\lambda(t) \, d\lambda(x).
\]

We have,

\[
\int_{(0, \infty)} \int_{[1, 2]} e^{-xt} \, d\lambda(x) \, d\lambda(t) = \int_{(0, \infty)} \int_{[1, 2]} e^{-xt} \, dx \, d\lambda(t) = \int_{(0, \infty)} (e^{-t} - e^{-2t}) \frac{1}{t} \, d\lambda(t).
\]
Therefore, \( \int_{(0,\infty)} (e^{-t} - e^{-2t}) \frac{1}{t} d\lambda(t) = \log(2). \)

(4) Let \((X, \mathcal{A}, \mu)\) be a measure space, and \(u \in \mathcal{M}(\mathcal{A})\) satisfies \(u^n \in L^1(\mu)\) for all \(n \geq 1\).

(a) Suppose \( \lim_{n \to \infty} \int u^n d\mu \) exists and is finite. For \(k \geq 1\), let \(E_k = \{x \in X : |u(x)| > 1 + 1/k\} \).

Prove that \( \int u^{2n} d\mu \geq (1 + 1/k)^{2n} \mu(E_k) \) for all \(n \geq 1\). (0.75 pts)

(b) Let \(E = \{x \in X : |u(x)| > 1\} \). Prove that \( \mu(E) = 0 \) and conclude that \(|u(x)| \leq 1 \) \(\mu\) a.e. (Hint: give a proof by contradiction) (1.25 pts)

(c) Prove that \( \int u^n d\mu = c \) is a constant for all \(n \geq 1\) if and only if \(u = 1_A \mu\) a.e. for some measurable set \(A \in \mathcal{A}\). (Hint: consider the function \(u^2(1-u)^2\) (1 pt)

**Proof (a)** For any \(n \geq 1\),

\[
u^{2n} = u^{2n}1_{E_k} + u^{2n}1_{E_k^c} \geq u^{2n}1_{E_k} \geq (1 + 1/k)^{2n}1_{E_k}.
\]

Thus, for all \(n \geq 1\)

\[
\int u^{2n} d\mu \geq (1 + 1/k)^{2n} \mu(E_k).
\]

**Proof (b)** Assume for the sake of getting a contradiction that \(\mu(E) > 0\). Note the sequence \((E_k)_{k \in \mathbb{N}}\) as given in part (a), is an increasing sequence of measurable sets with \(E = \bigcup_{k=1}^{\infty} E_k\). Since \(\mu(E) > 0\), then there exists \(k \geq 1\) sufficiently large such that \(\mu(E_k) > 0\). From part (a), we have for all \(n \geq 1\)

\[
\int u^{2n} d\mu \geq (1 + 1/k)^{2n} \mu(E_k).
\]

This implies that

\[
\lim_{n \to \infty} \int u^{2n} d\mu \geq \lim_{n \to \infty} (1 + 1/k)^{2n} \mu(E_k) = \infty,
\]

contradicting the fact that \(\lim_{n \to \infty} \int u^n d\mu < \infty\). Thus \(\mu(E) = 0\) and \(|u(x)| \leq 1 \) \(\mu\) a.e.

**Proof (c)** If \(u = 1_A\) for some measurable set \(A \in \mathcal{A}\), then \(u^n = 1_A\) for all \(n \geq 1\) and hence

\[
\int u^n d\mu = \mu(A) < \infty\] for all \(n \geq 1\). Therefore, the result holds with \(c = \mu(A)\).

Conversely, assume \(\int u^n d\mu = c\) is a constant for all \(n \geq 1\). Note that

\[
\int u^2(1-u)^2 d\mu = \int u^2 d\mu - 2 \int u^3 d\mu + \int u^4 d\mu = c - 2c + c = 0.
\]

Since \(u^2(1-u)^2 \geq 0\), this implies that \(u^2(1-u)^2 = 0 \) \(\mu\) a.e. implying that \(u\) takes only two values 0 and 1 \(\mu\) a.e. Equivalently, \(u\) is \(\mu\) a.e.equals the indicator function \(1_A\) with \(A = \{x \in X : u(x) = 1\}\).