Measure and Integration: Final 2019-20

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra and λ is Lebesgue measure. For $n \geq 1$, let $u_n(x) = \mathbb{I}_{[0,1-2^{-n})}(x)\cos(e^{-x/n})\, x^2$.

(a) Prove that $\lim_{n\to\infty} \int u_n \, d\lambda = \frac{1}{3}$. (1 pt)

(b) Let $1 < p < \infty$, prove that $\Big| \sum_{n=1}^{\infty} \left(\frac{u_n}{n}\right)^p \Big| < \infty$ μ a.e. (1 pt)

(2) Let $(X, \mathcal{A}, \mu)$ be a **finite** measure space, and $1 < p, q < \infty$ two conjugate numbers (i.e. $1/p + 1/q = 1$). Let $v \in \mathcal{M}(\mathcal{A})$ be a measurable function satisfying

$$\int |uv| \, d\mu \leq ||u||_q$$

for all $u \in \mathcal{L}^q(\mu)$.

(a) For $n \geq 1$, let $A_n = \{x \in X : |v(x)| \leq n\}$ and $v_n = \mathbb{I}_{A_n}|v|^{p/q}$. Prove that $v_n \in \mathcal{L}^q(\mu)$ and that $||v_n||_q^q = ||\mathbb{I}_{A_n} v||_p^p$ for all $n \geq 1$. (0.75 pt)

(b) Prove that $||\mathbb{I}_{A_n} v||_p \leq 1$ for all $n \geq 1$. (1.5 pt)

(c) Prove that $v \in \mathcal{L}^p(\mu)$. (0.75 pt)

(3) Consider the product space $([1,2], \mathcal{B}([1,2]) \otimes \mathcal{B}((0,\infty)\lambda \otimes \lambda)$ with λ is Lebesgue measure restricted on the appropriate spaces. Consider the fuction $f : [1,2] \times (0,\infty) \to (0,\infty)$ defined by $f(x,t) = e^{-xt}$.

(a) Prove that $f \in \mathcal{L}^1(\lambda \otimes \lambda)$. (1pt)

(b) Prove that $\int_{(0,\infty)} (e^{-t} - e^{-2t}) \frac{1}{t} d\lambda(t) = \log(2)$. (1pt)

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{M}(\mathcal{A})$ satisfies $u^n \in \mathcal{L}^1(\mu)$ for all $n \geq 1$.

(a) Suppose $\lim_{n\to\infty} \int u^n \, d\mu$ exists and is finite. For $k \geq 1$, let $E_k = \{x \in X : |u(x)| > 1 + 1/k\}$. Prove that $\int u^{2n} \, d\mu \geq (1 + 1/k)^{2n} \mu(E_k)$ for all $n \geq 1$. (0.75 pt)

(b) Let $E = \{x \in X : |u(x)| > 1\}$. Prove that $\mu(E) = 0$ and conclude that $|u(x)| \leq 1$ μ a.e. (Hint: give a proof by contradiction) (1.25 pt)

(c) Prove that $\int u^n \, d\mu = c$ is a constant for all $n \geq 1$ **if and only if** $u = \mathbb{I}_A$ μ a.e. for some measurable set $A \in \mathcal{A}$. (Hint: consider the function $u^2(1-u)^2$) (1 pt)