Measure and Integration: Solutions Retake Exam 2019-20

(1) Consider the measure space $\Big((0,1), \mathcal{B}((0,1)), \lambda\Big)$, where $\mathcal{B}((0,1))$ is the Borel σ-algebra restricted to the interval $(0,1)$ and λ is the restriction of Lebesgue measure to $(0,1)$. Let $u \in \mathcal{L}^1((0,1))$ be **non-negative** and **monotonically increasing**.

(a) Prove that $\lim_{n\to\infty} \int_{(0,1)} u(x^n)\, d\lambda(x) = \inf_{y\in(0,1)} u(y)$. (1 pt)

(b) Prove that $\lim_{n\to\infty} \int_{(0,1)} x^n u(x)\, d\lambda(x) = 0$. (0.5 pt)

Proof(a): For any $x \in (0,1)$, $x^n \searrow 0$. Since u is non-negative and increasing, we see that $\{u(x^n)\}_{n\in\mathbb{N}}$ is decreasing and

$$\lim_{n\to\infty} u(x^n) = \inf_{n\in N} u(x^n) = \inf_{y\in(0,1)} u(y).$$

Since $\inf_{n\in\mathbb{N}} \int u(x^n)\, d\lambda(x) \geq 0 > -\infty$, by Monotone Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \int_{(0,1)} u(x^n)\, d\lambda(x) &= \inf_{n\in\mathbb{N}} \int_{(0,1)} u(x^n)\, d\lambda(x)\\ &= \int_{(0,1)} \inf_{n\in\mathbb{N}} u(x^n)\, d\lambda(x)\\ &= \int_{(0,1)} \inf_{y\in(0,1)} u(y)\, d\lambda(x)\\ &= \inf_{y\in(0,1)} u(y),\end{aligned}$$

The last equality follows from the fact that $\inf_{y\in(0,1)} u(y)$ is a constant and λ is a probability measure.

Proof(b): For each $x \in (0,1)$ and for every $n \geq 1$, we have $0 \leq x^n u(x) < u(x)$. Since $u \in \mathcal{L}((0,1))$, and $\lim_{n\to\infty} x^n u(x) = 0$ for all $x \in (0,1)$ (note the $u(x) < \infty$), we have by Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \int_{(0,1)} x^n u(x)\, d\lambda(x) = \int_{(0,1)} \lim_{n\to\infty} x^n u(x)\, d\lambda(x) = 0.$$

(2) Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the restriction of the Borel σ-algebra on $[0,\infty)$, and λ is the restriction of Lebesgue measure on $\mathcal{B}([0,\infty))$. Define $f : [0,\infty) \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbf{1}_{[n-1,n)}(x).$$

(a) Show that f is $\mathcal{B}([0,\infty))/\mathcal{B}(\mathbb{R})$ measurable and that $f \notin \mathcal{L}^1(\lambda)$. (1 pt)

(b) Show that the improper Riemann-integral $(R)\int_0^\infty f(x)\, dx = \lim_{n\to\infty} (R)\int_0^n f(x)\, dx$ exists and is finite. Give an argument why parts (a) and (b) do not contradict Corollary 12.11 (Corollary 11.9 in edition 1). (0.5 pt)

Proof(a): Define $f_m = \sum_{n=1}^{m} \frac{(-1)^n}{n} \mathbf{1}_{[n-1,n)}(x)$ for $m \geq 1$ and $x \in [0, \infty)$. Since $[n-1, n) \in \mathcal{B}([0, \infty))$, then f_m is a sequence of measurable simple functions with $f = \lim_{m\to\infty} f_m = \sup_{m\geq 1} f_m$. So f is the limit of measurable functions, and by Corllary 8.10 (Corollary 8.9 in edition 1), f is measurable. Notice that $|f_m| \nearrow |f|$, so by Beppo-Levi we have

$$\int |f| \, d\lambda = \sup_m \int |f_m| \, d\lambda = \sup_m \sum_{n=1}^{m} \frac{1}{n} = \infty.$$

Thus, $f \notin \mathcal{L}^1(\lambda)$.

Proof(b): For any $n \geq 1$, we have $f\mathbf{1}_{[0,n)} = \sum_{m=1}^{n} \frac{(-1)^m}{m} \mathbf{1}_{[m-1,m)}$. Hence

$$(R) \int_0^\infty f(x) \, dx = \lim_{n\to\infty} (R) \int_0^n f(x) \, dx = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{(-1)^m}{m} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} = -\ln 2 < \infty.$$

Since condition (12.10) ((11.10) in edition 1) does not hold, Corollary 12.11 (Corollary 11.9 in edition 1) does not apply.

(3) Let $(X, \mathcal{B}, \mu)$ be a **probability** space and $T : X \to X$ a surjective $\mathcal{B}/\mathcal{B}$ measurable transformation such that $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{B}$.

(a) Prove that for any $u \in \mathcal{L}^1(\mu)$, one has

$$\int u \, d\mu = \int u \circ T \, d\mu.$$

(1 pt)

(b) Suppose that $u \in \mathcal{L}^1(\mu)$ satisfies $u \leq u \circ T$ μ a.e. Prove that $u = u \circ T$ μ a.e. (0.5 pt)

(c) Let $u \in \mathcal{L}^1(\mu)$ and $\epsilon > 0$. For $n \geq 1$, let $T^n = T \circ T \circ \cdots \circ T$ denotes the n-fold composition of T with itself. Define for $n \geq 1$,

$$A_{n,\epsilon} = \{x \in X : \frac{|u(T^n(x))|}{n^2} > \epsilon\}.$$

Prove that $\sum_{n=1}^{\infty} \mu(A_{n,\epsilon}) < \infty$. (1 pt)

(d) Using the same notation as in part (c), prove that $\mu\Big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_{n,\epsilon}\Big) = 0$. (1 pt)

(e) Let $u \in \mathcal{L}^1(\mu)$, define $Y = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A^c_{n,1/k}$ (we are using the same notation as in the previous parts). Prove that $\mu(Y) = 1$, and that for any $x \in Y$ one has $\lim_{n\to\infty} \frac{|u(T^n(x))|}{n^2} = 0$. Conclude that $\lim_{n\to\infty} \frac{|u(T^n(x))|}{n^2} = 0$ μ a.e. (1 pt)

Proof (a): We use the standard argument. For $u = \mathbb{I}_A$ with $A \in \mathcal{B}$, we have

$$\int u \, d\mu = \int \mathbb{I}_A \, d\mu = \mu(A) = \mu(T^{-1}(A) = \int \mathbb{I}_{T^{-1}(A)} \, d\mu = \int \mathbb{I}_A \circ T \, d\mu = \int u \circ T \, d\mu.$$

Now assume $u = \sum_{n=1}^{m} a_n \mathbb{I}_{A_m}$ with $a_n \geq 0$ and $A_m \in \mathcal{B}$, i.e. u is a non-negative measurable simple function. We have from the above and the linearity of the integral,

$$\begin{aligned} \int u \, d\mu = \int \sum_{n=1}^{m} a_n \mathbb{I}_{A_m} &= \sum_{n=1}^{m} a_n \int \mathbb{I}_{A_m} \, d\mu \\ &= \sum_{n=1}^{m} a_n \int \mathbb{I}_{A_m} \circ T \, d\mu \\ &= \int \sum_{n=1}^{m} a_n \, \mathbb{I}_{A_m} \circ T \, d\mu \\ &= \int u \circ T \, d\mu. \end{aligned}$$

Now, let u be a non-negative integrable function. By Theorem 8.8, there exists a sequence $\{f_n\}$ of non-negative measurable increasing functions such that $u = \sup_n f_n$. By Beppo-Levi and the fact that $f_n \circ T \nearrow u \circ T$, we have

$$\int u \, d\mu = \sup_n \int f_n \, d\mu = \sup_n \int f_n \circ T \, d\mu = \int \sup_n f_n \circ T \, d\mu = \int u \circ T \, d\mu.$$

Finally, let $u \in \mathcal{L}^1(\mu)$, write $u = u^+ - u^-$. Note that u^+, u^- are non-negative integrable functions, hence by the above we have

$$\int u \, d\mu = \int u^+ \, d\mu - \int u^- \, d\mu = \int u^+ \circ T \, d\mu - \int u^- \circ T \, d\mu = \int u \circ T \, d\mu.$$

Proof (b): Note that $u \circ T - u$ is integrable and $u \circ T - u \geq 0$ μ a.e. Thus,

$$\int (u \circ T - u) \, d\mu = \int u \, d\mu - \int u \circ T \, d\mu = \int u \, d\mu - \int u \, d\mu = 0.$$

By Theorem 11.2(i), we have $u \circ T - u = 0$ μ a.e. i.e. $u \circ T = u$ μ a.e.

Proof (c): Part (a) applied repeatedly gives $u \circ T^n \in \mathcal{L}^1(\mu)$ and $\int u \, d\mu = \int u \circ T^n \, d\mu$ for all $n \geq 1$. By Markov inequality, we have

$$\mu(A_{n,\epsilon}) = \mu\Big(\{x \in X : \frac{|u(T^n(x))|}{n^2} > \epsilon\}\Big) \leq \frac{1}{n^2 \epsilon} \int u \circ T^n \, d\mu = \frac{1}{n^2 \epsilon} \int u \, d\mu.$$

Since $|\int u \, d\mu| < \infty$, we have

$$\sum_{n=1}^{\infty} \mu(A_{n,\epsilon}) \leq \Big(\int u \, d\mu\Big) \sum_{n=1}^{\infty} \frac{1}{n^2 \epsilon} < \infty.$$

Proof (d): First note that $\Big\{\bigcup_{n=m}^{\infty} A_{n,\epsilon}\Big\}_{m \in \mathbb{N}}$ is a decreasing sequence of measurable sets and μ is a probability measure. Thus, by Proposition 4.3(vii), we have

$$\begin{aligned} \mu\Big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_{n,\epsilon}\Big) &= \lim_{m \to \infty} \mu\Big(\bigcup_{n=m}^{\infty} A_{n,\epsilon}\Big) \\ &\leq \lim_{m \to \infty} \sum_{n=m}^{\infty} \mu(A_{n,\epsilon}) \\ &\leq \Big(\int u \, d\mu\Big) \lim_{m \to \infty} \sum_{n=m}^{\infty} \frac{1}{n^2 \epsilon} = 0. \end{aligned}$$

Proof (e): Since μ is a probability measure, by part (d) we have for any $k \geq 1$,

$$\mu\Big(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_{n,1/k}\Big) = 0,$$

so that

$$\mu\Big(\bigcup_{k=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_{n,1/k}\Big) = 0.$$

Thus, $Y = \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} A^c_{n,1/k}$, is a measurable set (since $u \circ T^n$ is a measurable function) with $\mu(Y) = 1$. Now, for $x \in Y$, we have for any $k \geq 1$, there exists $m \geq 1$ such that for all $n \geq m$, we have $\frac{|u(T^n(x))|}{n^2} \leq \frac{1}{k}$. This implies that $\lim_{n\to\infty} \frac{|u(T^n(x))|}{n^2} = 0$ for all $x \in Y$. Since $\mu(Y) = 1$, we conclude that $\lim_{n\to\infty} \frac{|u(T^n(x))|}{n^2} = 0$ μ a.e.

(4) Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the Borel σ-algebra, and λ is Lebesgue measure on $[0,\infty)$. Let $f(x,y) = ye^{-(1+x^2)y^2}$ for $0 \leq x, y < \infty$.

  (a) Show that $f \in \mathcal{L}^1(\lambda \times \lambda)$, and determine the value of $\int_{[0,\infty)\times[0,\infty)} f\, d(\lambda\times\lambda)$. (1 pt)

  (b) Prove that $\int_{[0,\infty)\times[0,\infty)} f\, d(\lambda\times\lambda) = \Big(\int_{[0,\infty)} e^{-x^2}\, d\lambda(x)\Big)^2$. Use part (a) to deduce the value of $\int_{[0,\infty)} e^{-x^2}\, d\lambda(x)$. (1.5 pt)

Proof(a): The function f is non-negative and continuous, and hence measurable. For each fixed $x \geq 0$, the improper Riemann-integral of the function $y \to ye^{-(1+x^2)y^2}$ exists, hence

$$\int_{[0,\infty)} ye^{-(1+x^2)y^2}\, d\lambda(y) = (R)\int_0^{\infty} ye^{-(1+x^2)y^2}\, dy = \frac{1}{2(1+x^2)}.$$

Similarly the improper Riemann-integral of the function $x \to \frac{1}{2(1+x^2)}$ exists and hence

$$\int_{[0,\infty)} \frac{1}{2(1+x^2)}\, d\lambda(x) = (R)\int_0^{\infty} \frac{1}{2(1+x^2)}\, dx = \frac{\pi}{4}.$$

By Tonelli's Theorem

$$\int_{[0,\infty)\times[0,\infty)} f\, d(\lambda\times\lambda) = \int_{[0,\infty)}\int_{[0,\infty)} ye^{-(1+x^2)y^2}\, d\lambda(y)\, d\lambda(x) = \frac{\pi}{4}.$$

This implies that $f \in \mathcal{L}^1(\lambda\times\lambda)$, and the integral has value $\frac{\pi}{4}$.

Proof(b): First note that with a simple substitution $u = xy$, one has by Theorem 7.10 that

$$\int_{[0,\infty)} ye^{-(1+x^2)y^2}\, d\lambda(x) = \int_{[0,\infty)} e^{-y^2}e^{-u^2}\, d\lambda(u).$$

Hence,

$$\int_{[0,\infty)}\int_{[0,\infty)} ye^{-(1+x^2)y^2}\, d\lambda(x)\, d\lambda(y) = \Big(\int_{[0,\infty)} e^{-y^2}\, d\lambda(y)\Big)^2.$$

By Tonelli's Theorem we have

$$\int_{[0,\infty)\times[0,\infty)} f\, d(\lambda\times\lambda) = \int_{[0,\infty)}\int_{[0,\infty)} ye^{-(1+x^2)y^2}\, d\lambda(x)\, d\lambda(y) = \Big(\int_{[0,\infty)} e^{-y^2}\, d\lambda(y)\Big)^2.$$

From part (a), we have

$$\Big(\int_{[0,\infty)} e^{-y^2}\, d\lambda(y)\Big)^2 = \frac{\pi}{4},$$

hence

$$\int_{[0,\infty)} e^{-y^2}\, d\lambda(y) = \sqrt{\frac{\pi}{4}}.$$