Measure and Integration: Retake Exam 2019-20

(1) Consider the measure space $\Big((0,1), \mathcal{B}((0,1)), \lambda\Big)$, where $\mathcal{B}((0,1))$ is the Borel σ-algebra restricted to the interval $(0,1)$ and λ is the restriction of Lebesgue measure to $(0,1)$. Let $u \in \mathcal{L}^1((0,1))$ be **non-negative** and **monotonically increasing**.

(a) Prove that $\lim_{n\to\infty} \int_{(0,1)} u(x^n)\, d\lambda(x) = \inf_{y\in(0,1)} u(y)$. (1 pt)

(b) Prove that $\lim_{n\to\infty} \int_{(0,1)} x^n u(x)\, d\lambda(x) = 0$. (0.5 pt)

(2) Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the restriction of the Borel σ-algebra on $[0,\infty)$, and λ is the restriction of Lebesgue measure on $\mathcal{B}([0,\infty))$. Define $f : [0,\infty) \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbf{1}_{[n-1,n)}(x).$$

(a) Show that f is $\mathcal{B}([0,\infty))/\mathcal{B}(\mathbb{R})$ measurable and that $f \notin \mathcal{L}^1(\lambda)$. (1 pt)

(b) Show that the improper Riemann-integral $(R)\int_0^\infty f(x)\,dx = \lim_{n\to\infty} (R)\int_0^n f(x)\,dx$ exists and is finite. Give an argument why parts (a) and (b) do not contradict Corollary 12.11 (Corollary 11.9 in edition 1). (0.5 pt)

(3) Let $(X, \mathcal{B}, \mu)$ be a **probability** space and $T : X \to X$ a surjective $\mathcal{B}/\mathcal{B}$ measurable transformation such that $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{B}$.

(a) Prove that for any $u \in \mathcal{L}^1(\mu)$, one has

$$\int u\, d\mu = \int u \circ T\, d\mu.$$

(1 pt)

(b) Suppose that $u \in \mathcal{L}^1(\mu)$ satisfies $u \le u \circ T$ μ a.e. Prove that $u = u \circ T$ μ a.e. (0.5 pt)

(c) Let $u \in \mathcal{L}^1(\mu)$ and $\epsilon > 0$. For $n \ge 1$, let $T^n = T \circ T \circ \cdots \circ T$ denotes the n-fold composition of T with itself. Define for $n \ge 1$,

$$A_{n,\epsilon} = \{x \in X : \frac{|u(T^n(x))|}{n^2} > \epsilon\}.$$

Prove that $\sum_{n=1}^{\infty} \mu(A_{n,\epsilon}) < \infty$. (1 pt)

(d) Using the same notation as in part (c), prove that $\mu\Big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_{n,\epsilon}\Big) = 0$. (1 pt)

(e) Let $u \in \mathcal{L}^1(\mu)$, define $Y = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A^c_{n,1/k}$ (we are using the same notation as in the previous parts). Prove that $\mu(Y) = 1$, and that for any $x \in Y$ one has $\lim_{n\to\infty} \frac{|u(T^n(x))|}{n^2} = 0$. Conclude that $\lim_{n\to\infty} \frac{|u(T^n(x))|}{n^2} = 0$ μ a.e. (1 pt)

(4) Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the Borel σ-algebra, and λ is Lebesgue measure on $[0,\infty)$. Let $f(x,y) = ye^{-(1+x^2)y^2}$ for $0 \leq x, y < \infty$.

(a) Show that $f \in \mathcal{L}^1(\lambda \times \lambda)$, and determine the value of $\int_{[0,\infty)\times[0,\infty)} f \, d(\lambda \times \lambda)$. (1 pt)

(b) Prove that $\int_{[0,\infty)\times[0,\infty)} f \, d(\lambda \times \lambda) = \left(\int_{[0,\infty)} e^{-x^2} \, d\lambda(x) \right)^2$. Use part (a) to deduce the value of $\int_{[0,\infty)} e^{-x^2} \, d\lambda(x)$. (1.5 pt)