Measure and Integration: Solutions Mid-Term, 2019-20

(1) Let \( X \) be a set. We call collection \( \mathcal{F} \) of subsets of \( X \) an algebra if the following conditions hold:

(i) \( \emptyset \in \mathcal{F} \), (ii) if \( A \in \mathcal{F} \), then \( A^c \in \mathcal{F} \), and (iii) if \( A, B \in \mathcal{F} \), then \( A \cup B \in \mathcal{F} \).

(a) Let \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \) be a strictly increasing sequence of algebras on \( X \). Show that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is an algebra on \( X \). (1 pt)

(b) Let \( \mu \) be a pre-measure on \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \). Find a measure \( \nu \) on \( \sigma\left( \bigcup_{n=1}^{\infty} \mathcal{F}_n \right) \) extending \( \mu \), i.e. \( \mu(A) = \nu(A) \) for all \( A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \). (1 pts)

**Proof (a):** We check the three conditions mentioned above. Clearly \( \emptyset \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \) since \( \emptyset \in \mathcal{F}_n \) for all \( n \). Now, let \( A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \), then there exists an \( n \) such that \( A \in \mathcal{F}_n \). Since \( \mathcal{F}_n \) is an algebra, we have \( A^c \in \mathcal{F}_n \) which implies that \( A^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \). Finally, let \( A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \), then there exist \( n, m \) such that \( A \in \mathcal{F}_n \) and \( B \in \mathcal{F}_m \). Assume with no loss of generality that \( n \leq m \), then \( \mathcal{F}_n \subseteq \mathcal{F}_m \) and hence \( A, B \in \mathcal{F}_m \). Since \( \mathcal{F}_m \) is an algebra, we have that \( A \cup B \in \mathcal{F}_m \) implying that \( A \cup B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \). Therefore \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is an algebra.

**Proof (b):** We first show that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is a semi-ring (in fact this is true for any algebra). So let \( A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \). By properties (ii) and (iii) of an algebra, we have

\[
A \cap B = \left( A^c \cup B^c \right)^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n,
\]

and

\[
A \setminus B = A \cap B^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n.
\]

This shows that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is a semi-ring. If \( \mu \) is a pre-measure on \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \), then by Carathéodory Extension Theorem, the set function \( \nu \) defined on \( \sigma\left( \bigcup_{n=1}^{\infty} \mathcal{F}_n \right) \) by

\[
\nu(B) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) : A_n \in \bigcup_{n=1}^{\infty} \mathcal{F}_n, n \in \mathbb{N}, B \subseteq \bigcup_{n=1}^{\infty} A_n \right\}
\]

is a measure on \( \sigma\left( \bigcup_{n=1}^{\infty} \mathcal{F}_n \right) \) extending \( \mu \).

(2) Let \((X, \mathcal{A}, \mu)\) be a measure space. Consider the collection

\[
\mathcal{C} = \left\{ C \subseteq X : \forall \epsilon > 0, \exists A_\epsilon, B_\epsilon \in \mathcal{A} \text{ such that } A_\epsilon \subseteq C \subseteq B_\epsilon \text{ and } \mu(A_\epsilon \Delta B_\epsilon) < \epsilon \right\}.
\]
(a) Prove that $\mathcal{C}$ is a $\sigma$-algebra on $X$ containing $\mathcal{A}$. (2 pts)

(b) Define $\nu$ on $\mathcal{C}$ by $\nu(C) = \inf \{ \mu(B) : C \subseteq B \text{ and } B \in \mathcal{A} \}$. Prove that $\nu$ is an outer measure on $\mathcal{C}$ extending $\mu$ on $\mathcal{A}$, i.e. $\nu(\emptyset) = 0$, $\nu$ is monotone and $\sigma$-subadditive, and $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$. (2 pts)

**Proof (a):** It is easy to see that $A \subseteq \mathcal{C}$, since for any $D \in \mathcal{A}$ and any $\epsilon > 0$, one can take $A = B_\epsilon = D$. We now check that $\mathcal{C}$ is a $\sigma$-algebra. Since $\emptyset \in \mathcal{A}$, it follows that $\emptyset \in \mathcal{C}$. Now, let $C \in \mathcal{C}$ and $\epsilon > 0$. Then, there exist $A_\epsilon, B_\epsilon \in \mathcal{A}$ such that $A_\epsilon \subseteq C \subseteq B_\epsilon$ and $\mu(A_\epsilon, B_\epsilon) < \epsilon$. Thus, $A_\epsilon \in \mathcal{C}$ and $B_\epsilon \in \mathcal{C}$ since $\mathcal{C}$ is a $\sigma$-algebra on $X$ containing $\mathcal{A}$. Finally, let $(C_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}$ and let $\epsilon > 0$. For each $n \in \mathbb{N}$, there exist $A_{\epsilon,n}, B_{\epsilon,n} \in \mathcal{A}$ such that $A_{\epsilon,n} \subseteq C_n \subseteq B_{\epsilon,n}$ and $\mu(A_{\epsilon,n}, B_{\epsilon,n}) < \epsilon/2^n$. Let $A_\epsilon = \bigcup_{n=1}^\infty A_{\epsilon,n}$ and $B_\epsilon = \bigcup_{n=1}^\infty B_{\epsilon,n}$, then $A_\epsilon, B_\epsilon \in \mathcal{C}$ and $A_\epsilon \subseteq \bigcup_{n=1}^\infty C_n \subseteq B_\epsilon$. It is easy to check that $A_\epsilon, B_\epsilon \in \mathcal{C}$ and $A_\epsilon \Delta B_\epsilon \subseteq \bigcup_{n=1}^\infty A_{\epsilon,n} \Delta B_{\epsilon,n}$.

Thus, $\mu(A_\epsilon \Delta B_\epsilon) \leq \sum_{n=1}^\infty \mu(A_{\epsilon,n} \Delta B_{\epsilon,n}) \leq \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon$.

This shows that $\bigcup_{n=1}^\infty C_n \in \mathcal{C}$. Thus, $\mathcal{C}$ is a $\sigma$-algebra on $X$ containing $\mathcal{A}$.

**Proof (b):** We first show that $\nu$ extends $\mu$. So let $A \in \mathcal{A}$, since $A \subseteq A$, we see from the definition of $\nu$ that $\nu(A) \leq \mu(A)$. On the other hand, for any $B \in \mathcal{A}$ such that $A \subseteq B$, we have by monotonicity of $\mu$ that $\mu(A) \leq \mu(B)$. Thus,

$$\mu(A) \leq \inf \{ \mu(B) : A \subseteq B \text{ and } B \in \mathcal{A} \} = \nu(A).$$

This shows that $\nu(A) = \mu(A)$ for any $A \in \mathcal{A}$. We now check that $\nu$ is an outer measure. Since $\emptyset \in \mathcal{A}$, and $\mu$ and $\nu$ agree on $\mathcal{A}$, we have $\nu(\emptyset) = \mu(\emptyset) = 0$. Let $C, D \in \mathcal{C}$ with $C \subseteq D$. Since

$$\{ B \in \mathcal{A} : D \subseteq B \} \subseteq \{ B \in \mathcal{A} : C \subseteq B \},$$

we have

$$\nu(C) = \inf \{ \mu(B) : C \subseteq B, B \in \mathcal{A} \} \leq \inf \{ \mu(B) : D \subseteq B, B \in \mathcal{A} \} = \nu(D).$$

Thus, $\nu$ is monotone. Finally, let $(C_n)_{n \in \mathbb{N}}$ be any countable collection in $\mathcal{C}$ and $\epsilon > 0$. By the definition of the infimum, we can find for each $n \in \mathbb{N}$, a set $B_n \in \mathcal{A}$ such that $C_n \subseteq B_n$ and $\mu(B_n) \leq \nu(C_n) + \epsilon/2^n$. Since $\mathcal{A}$ is a $\sigma$-algebra, we have that $\bigcup_{n=1}^\infty B_n \in \mathcal{A}$ and $\bigcup_{n=1}^\infty C_n \subseteq \bigcup_{n=1}^\infty B_n$. Thus,

$$\nu\left( \bigcup_{n=1}^\infty C_n \right) \leq \mu\left( \bigcup_{n=1}^\infty B_n \right) \leq \sum_{n=1}^\infty \mu(B_n) \leq \sum_{n=1}^\infty \nu(C_n) + \epsilon/2^n = \sum_{n=1}^\infty \nu(C_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\nu\left( \bigcup_{n=1}^\infty C_n \right) \leq \sum_{n=1}^\infty \nu(C_n)$ and $\nu$ is $\sigma$-subadditive. Therefore, $\nu$ is an outer measure.

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra, and $\lambda$ is Lebesgue measure. Define a function $u : \mathbb{R} \to \mathbb{R}$ by $u(x) = \sum_{n=0}^\infty \frac{x}{2^n} \mathbb{I}_{[n, n+1)}(x)$.

(a) Prove that $u$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. (0.5 pts)
(b) For $m \geq 0$, let $u_m(x) = \frac{x}{2^m} \cdot \mathbb{1}_{[m,m+1)}(x)$. Give an explicit sequence $(f_n^{(m)})_{n \in \mathbb{N}}$ of $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable simple functions with $f_n^{(m)} \not\rightarrow u_m$, i.e. $(f_n^{(m)})$ is an increasing sequence and $u_m(x) = \sup_{n \geq 1} f_n^{(m)}(x) = \lim_{n \to \infty} f_n^{(m)}(x)$ for all $x \in \mathbb{R}$. (1.5 pts)

(c) Use part (b) to prove that

$$\int u \, d\lambda = \sum_{n=0}^{\infty} \frac{2n+1}{2^{n+1}} = 3.$$

(Recall that $\sum_{n=0}^{m} k = \frac{m(m+1)}{2}$, $\sum_{n=0}^{\infty} r^n = \frac{1}{r-1}$ and $\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(r-1)^2}$, for $-1<r<1$) (2 pts)

**Proof (a):** The (identity) function $x \to x$ is continuous, and hence Borel measurable i.e. $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable (in fact also $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, but since all the functions under consideration have finite values, we can restrict our attention to $\mathcal{B}(\mathbb{R})$). Since $[n,n+1) \in \mathcal{B}(\mathbb{R})$, it follows that $\frac{n}{2n} \mathbb{1}_{[n,n+1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. Thus for any $n \geq 0$, the product $\frac{x}{2n} \mathbb{1}_{[n,n+1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, and also the sum $\sum_{k=0}^{n} \frac{x}{2k} \mathbb{1}_{[k,k+1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. Since

$$u(x) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x}{2k} \mathbb{1}_{[k,k+1)} = \sup_{n} \sum_{k=0}^{n} \frac{x}{2k} \mathbb{1}_{[k,k+1)};$$

we have by Corollary 8.10 that $u$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable.

**Proof (b):** We apply the recipe given in the proof of Theorem 8.8 (sombrero lemma). Define $f_n^{(m)}(x) = \sum_{k=0}^{2^n-1} \frac{1}{2^n} (m + \frac{k}{2^n}) \cdot \mathbb{1}_{[m+\frac{k}{2^n},m+\frac{k+1}{2^n})}(x)$. Since $[m+k/2^n,m+k+1/2^n) \in \mathcal{B}(\mathbb{R})$ for all $k$, then $f_n^{(m)}$ is a $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable simple function. We claim that $0 \leq f_n^{(m)} \leq f_{n+1}^{(m)} \leq u_m$ and $\lim_{n \to \infty} f_n^{(m)} = u_m$. First note that if $x \notin [m,m+1)$, then $f_n^{(m)}(x) = f_{n+1}^{(m)}(x) = u_m(x) = 0$ for all $n$. So assume $x \in [m,m+1)$ and fix any $n$. There exists $k \in \{0,1,\ldots,2^n-1\}$ such that $x \in [m+\frac{k}{2^n},m+\frac{k+1}{2^n})$. Since

$$[m+k/2^n,m+k+1/2^n) = [m+2k/2^{n+1},m+2k+1/2^{n+1}) \cup [m+2k+1/2^{n+1},m+2k+2/2^{n+1})$$

we consider two cases. If $x \in [m+2k/2^{n+1},m+2k+1/2^{n+1})$, then

$$f_n^{(m)}(x) = f_{n+1}^{(m)}(x) = \frac{1}{2^n} (m + \frac{k}{2^n}) \leq \frac{x}{2^n} = u_m(x), \text{ and } |u_m(x) - f_n^{(m)}(x)| \leq 2^{-n}.$$ 

If $x \in [m+2k+1/2^{n+1},m+2k+2/2^{n+1})$, then

$$f_n^{(m)}(x) = \frac{1}{2^n} (m + \frac{k}{2^n}) < \frac{1}{2^n} (m + \frac{2k+1}{2^{n+1}}) = f_{n+1}^{(m)}(x) \leq \frac{x}{2^n} = u_m(x),$$

and $|u_m(x) - f_n^{(m)}(x)| \leq 2^{-n}$. Thus, $f_n^{(m)} \not\rightarrow u_m$.

**Proof (c):** Note that $u = \sum_{m=0}^{\infty} u_m$, thus by Corollary 9.9, we have

$$\int u \, d\lambda = \sum_{m=0}^{\infty} \int u_m \, d\lambda.$$
We calculate first $\int u_m \, d\lambda$ using Corollary 9.7

$$\int u_m \, d\lambda = \lim_{n \to \infty} \int f_n^{(m)} \, d\lambda$$

$$= \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \frac{1}{2^m} \left(1 + \frac{k}{2^n}\right) \cdot \lambda\left([m + \frac{k}{2^n}, m + \frac{k+1}{2^n}]\right)$$

$$= \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \frac{1}{2^m} \left(1 + \frac{k}{2^n}\right) \cdot \frac{1}{2^n}$$

$$= \lim_{n \to \infty} \frac{1}{2^m} \left(m + \frac{1}{4^n} \cdot \sum_{k=0}^{2^n-1} k\right)$$

$$= \lim_{n \to \infty} \frac{1}{2^m} \left(m + \frac{1}{4^n} \cdot \frac{2^n (2^n - 1)}{2}\right)$$

$$= \frac{1}{2^m} \left(m + \frac{1}{2}\right)$$

$$= \frac{2m + 1}{2^{m+1}}.$$

Using this we have,

$$\int u \, d\lambda = \sum_{m=0}^{\infty} \int u_m \, d\lambda$$

$$= \sum_{m=0}^{\infty} \frac{2m + 1}{2^{m+1}}$$

$$= \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{2^{m-1}} + \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{2^m}$$

$$= 3.$$