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Measure and Integration: Solutions Mid-Term, 2019-20

(1) Let X be a set. We call collection $\mathcal{F}$ of subsets of X an algebra if the following conditions hold:

(i) $\emptyset \in \mathcal{F}$, (ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and (iii) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

(a) Let $\mathcal{F}_1 \subset \mathcal{F}_1 \subset \cdots$ be a strictly increasing sequence of algebras on X. Show that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is an algebra on X. (1 pt)

(b) Let μ be a **pre**-measure on $\bigcup_{n=1}^{\infty} \mathcal{F}_n$. Find a measure ν on $\sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{F}_n\Big)$ extending μ, i.e. $\mu(A) = \nu(A)$ for all $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. (1 pts)

Proof (a): We check the three conditions mentioned above. Clearly $\emptyset \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$ since $\emptyset \in \mathcal{F}_n$ for all n. Now ,let $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, then there exists an n such that $A \in \mathcal{F}_n$. Since $\mathcal{F}_n$ is an algebra, we have $A^c \in \mathcal{F}_n$ which implies that $A^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Finally, let $A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, then there exist n, m such that $A \in \mathcal{F}_n$ and $B \in \mathcal{F}_m$. Assume with no loss of generality that $n \le m$, then $\mathcal{F}_n \subseteq \mathcal{F}_m$ and hence $A, B \in \mathcal{F}_m$. Since $\mathcal{F}_m$ is an algebra, we have that $A \cup B \in \mathcal{F}_m$ implying that $A \cup B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Therefore $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is an algebra.

Proof (b): We first show that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a semi-ring (in fact this is true for any algebra). So let $A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. By properties (ii) and (iii) of an algebra, we have

$$A \cap B = \Big(A^c \cup B^c\Big)^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n,$$

and

$$A \setminus B = A \cap B^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n.$$

This shows that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a semi-ring. If μ is a pre-measure on $\bigcup_{n=1}^{\infty} \mathcal{F}_n$, then by Carathéodory Extension Theorem, the set function ν defined on $\sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{F}_n\Big)$ by

$$\nu(B) = \inf\Big\{\sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \bigcup_{n=1}^{\infty} \mathcal{F}_n,\, n \in \mathbb{N}, B \subseteq \bigcup_{n=1}^{\infty} A_n\Big\}$$

is a measure on $\sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{F}_n\Big)$ extending μ.

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider the collection

$$\mathcal{C} = \Big\{C \subseteq X : \forall \epsilon > 0, \exists A_\epsilon, B_\epsilon \in \mathcal{A} \text{ such that } A_\epsilon \subseteq C \subseteq B_\epsilon \text{ and } \mu(A_\epsilon \Delta B_\epsilon) < \epsilon\Big\}.$$

(a) Prove that $\mathcal{C}$ is a σ-algebra on X containing $\mathcal{A}$. (2 pts)

(b) Define ν on $\mathcal{C}$ by $\nu(C) = \inf\Big\{\mu(B) : C \subseteq B \text{ and } B \in \mathcal{A}\Big\}$. Prove that ν is an **outer** measure on $\mathcal{C}$ extending μ on $\mathcal{A}$, i.e. $\nu(\varnothing) = 0$, ν is monotone and σ-subadditive, and $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$. (2 pts)

Proof (a): It is easy to see that $\mathcal{A} \subseteq \mathcal{C}$, since for any $D \in \mathcal{A}$ and any $\epsilon > 0$, one can take $A_\epsilon = B_\epsilon = D$. We now check that $\mathcal{C}$ is a σ-algebra. Since $\varnothing \in \mathcal{A}$, it follows that $\varnothing \in \mathcal{C}$. Now, let $C \in \mathcal{C}$ and $\epsilon > 0$. Then, there exist $A_\epsilon, B_\epsilon \in \mathcal{A}$ such that $A_\epsilon \subseteq C \subseteq B_\epsilon$ and $\mu(A_\epsilon \Delta B_\epsilon) < \epsilon$. Then, $A_\epsilon^c, B_\epsilon^c \in \mathcal{A}$ such that $B_\epsilon^c \subseteq C^c \subseteq A_\epsilon^c$, and $\mu(A_\epsilon^c \Delta B_\epsilon^c) = \mu(A_\epsilon \Delta B_\epsilon) < \epsilon$. Thus, $C^c \in \mathcal{C}$. Finally, let $(C_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ and let $\epsilon > 0$. For each $n \in N$, there exist $A_{\epsilon,n}, B_{\epsilon,n} \in \mathcal{A}$ such that $A_{\epsilon,n} \subseteq C_n \subseteq B_{\epsilon,n}$ and $\mu(A_{\epsilon,n}\Delta B_{\epsilon,n}) < \epsilon/2^n$. Let $A_\epsilon = \bigcup_{n=1}^{\infty} A_{\epsilon,n}$ and $B_\epsilon = \bigcup_{n=1}^{\infty} B_{\epsilon,n}$, then $A_\epsilon, B_\epsilon \in \mathcal{C}$ and $A_\epsilon \subseteq \bigcup_{n=1}^{\infty} C_n \subseteq B_\epsilon$. It is easy to check that $A_\epsilon \Delta B_\epsilon \subset \bigcup_{n=1}^{\infty} A_{\epsilon,n}\Delta B_{\epsilon,n}$. Thus,

$$\mu\Big(A_\epsilon \Delta B_\epsilon\Big) \leq \sum_{n=1}^{\infty} \mu\Big(A_{\epsilon,n}\Delta B_{\epsilon,n}\Big) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

This shows that $\bigcup_{n=1}^{\infty} C_n \in \mathcal{C}$. Thus, $\mathcal{C}$ is a σ-algebra on X containing $\mathcal{A}$.

Proof (b): We first show that ν extends μ. So let $A \in \mathcal{A}$, since $A \subseteq A$, we see from the definition of ν that $\nu(A) \leq \mu(A)$. On the otherhand, for any $B \in \mathcal{A}$ such that $A \subseteq B$, we have by monotonicity of μ that $\mu(A) \leq \mu(B)$. Thus,

$$\mu(A) \leq \inf\Big\{\mu(B) : A \subseteq B \text{ and } B \in \mathcal{A}\Big\} = \nu(A).$$

This shows that $\nu(A) = \mu(A)$ for any $A \in \mathcal{A}$. We now check that ν is an outer measure. Since $\varnothing \in \mathcal{A}$, and μ and ν agree on $\mathcal{A}$, we have $\nu(\varnothing) = \mu(\varnothing) = 0$. Let $C, D \in \mathcal{C}$ with $C \subseteq D$. Since

$$\Big\{B \in \mathcal{A} : D \subseteq B\Big\} \subseteq \Big\{B \in \mathcal{A} : C \subseteq B\Big\},$$

we have

$$\nu(C) = \inf\Big\{\mu(B) : C \subseteq B, B \in \mathcal{A}\Big\} \leq \inf\Big\{\mu(B) : D \subseteq B, B \in \mathcal{A}\Big\} = \nu(D).$$

Thus, ν is monotone. Finally, let $(C_n)_{n\in\mathbb{N}}$ be any countable collection in $\mathcal{C}$ and let $\epsilon > 0$. By the definition of the infimum, we can find for each $n \in \mathbb{N}$, a set $B_n \in \mathcal{A}$ such that $C_n \subseteq B_n$ and $\mu(B_n) \leq \nu(C_n) + \epsilon/2^n$. Since $\mathcal{A}$ is a σ-algebra, we have that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$ and $\bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{n=1}^{\infty} B_n$. Thus,

$$\begin{aligned}
\nu\Big(\bigcup_{n=1}^{\infty} C_n\Big) &\leq \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) \\
&\leq \sum_{n=1}^{\infty} \mu(B_n) \\
&\leq \sum_{n=1}^{\infty} \nu(C_n) + \epsilon/2^n \\
&= \sum_{n=1}^{\infty} \nu(C_n) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\nu\Big(\bigcup_{n=1}^{\infty} C_n\Big) \leq \sum_{n=1}^{\infty} \nu(C_n)$ and ν is σ-subadditive. Therefore, ν is an outer measure.

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ is Lebesgue measure. Define a function $u : \mathbb{R} \to \mathbb{R}$ by $u(x) = \sum_{n=0}^{\infty} \frac{x}{2^n} \cdot \mathbb{I}_{[n,n+1)}(x)$.

(a) Prove that u is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\overline{\mathbb{R}})$-measurable. (0.5 pts)

(b) For $m \geq 0$, let $u_m(x) = \frac{x}{2^m} \cdot \mathbb{I}_{[m,m+1)}(x)$. Give an explicit sequence $(f_n^{(m)})_{n\in\mathbb{N}}$ of $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable simple functions with $f_n^{(m)} \nearrow u_m$, i.e. $(f_n^{(m)})$ is an increasing sequence and $u_m(x) = \sup_{n\geq 1} f_n^{(m)}(x) = \lim_{n\to\infty} f_n^{(m)}(x)$ for all $x \in \mathbb{R}$. (1.5 pts)

(c) Use part (b) to prove that

$$\int u\, d\lambda = \sum_{n=0}^{\infty} \frac{2n+1}{2^{n+1}} = 3.$$

(Recall that $\sum_{n=0}^{m} k = \frac{m(m+1)}{2}$, $\sum_{n=0}^{\infty} r^n = \frac{1}{r-1}$ and $\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(r-1)^2}$, for $-1 < r < 1$) (2 pts)

Proof (a): The (identity) function $x \to x$ is continuous, and hence Borel measurable i.e. $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable (in fact also $\mathcal{B}(\overline{\mathbb{R}})/\mathcal{B}(\overline{\mathbb{R}})$-measurable, but since all the functions under consideration have finite values, we can restrict our attention to $\mathcal{B}(\mathbb{R})$). Since $[n, n+1) \in \mathcal{B}(\mathbb{R})$, it follows that $\frac{1}{2^n}\mathbb{I}_{[n,n+1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. Thus for any $n \geq 0$, the product $\frac{x}{2^n}\mathbb{I}_{[n,n+1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, and also the sum $\sum_{k=0}^{n} \frac{x}{2^k}\mathbb{I}_{[k,k+1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. Since

$$u(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x}{2^k}\mathbb{I}_{[k,k+1)} = \sup_{n} \sum_{k=0}^{n} \frac{x}{2^k}\mathbb{I}_{[k,k+1)},$$

we have by Corollary 8.10 that u is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable.

Proof (b): We apply the recipe given in the proof of Theorem 8.8 (sombrero lemma). Define $f_n^{(m)}(x) = \sum_{k=0}^{2^n-1} \frac{1}{2^m}\left(m + \frac{k}{2^n}\right) \cdot \mathbb{I}_{[m+\frac{k}{2^n}, m+\frac{k+1}{2^n})}(x)$. Since $\left[m + \frac{k}{2^n}, m + \frac{k+1}{2^n}\right) \in \mathcal{B}(\mathbb{R})$ for all k, then $f_n^{(m)}$ is a $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable simple function. We claim that $0 \leq f_n^{(m)} \leq f_{n+1}^{(m)} \leq u_m$ and $\lim_{n\to\infty} f_n^{(m)} = u_m$. First note that if $x \notin [m, m+1)$, then $f_n^{(m)}(x) = f_{n+1}^{(m)}(x) = u_m(x) = 0$ for all n. So assume $x \in [m, m+1)$ and fix any n. There exists $k \in \{0, 1, \cdots, 2^n - 1\}$ such that $x \in \left[m + \frac{k}{2^n}, m + \frac{k+1}{2^n}\right)$. Since

$$\left[m + \frac{k}{2^n}, m + \frac{k+1}{2^n}\right) = \left[m + \frac{2k}{2^{n+1}}, m + \frac{2k+1}{2^{n+1}}\right) \dot{\bigcup} \left[m + \frac{2k+1}{2^{n+1}}, m + \frac{2k+2}{2^{n+1}}\right),$$

we consider two cases. If $x \in \left[m + \frac{2k}{2^{n+1}}, m + \frac{2k+1}{2^{n+1}}\right)$, then

$$f_n^{(m)}(x) = f_{n+1}^{(m)}(x) = \frac{1}{2^m}\left(m + \frac{k}{2^n}\right) \leq \frac{x}{2^m} = u_m(x), \text{ and } |u_m(x) - f_n^{(m)}(x)| \leq 2^{-n}.$$

If $x \in \left[m + \frac{2k+1}{2^{n+1}}, m + \frac{2k+2}{2^{n+1}}\right)$, then

$$f_n^{(m)}(x) = \frac{1}{2^m}\left(m + \frac{k}{2^n}\right) < \frac{1}{2^m}\left(m + \frac{2k+1}{2^{n+1}}\right) = f_{n+1}^{(m)}(x) \leq \frac{x}{2^m} = u_m(x),$$

and $|u_m(x) - f_n^{(m)}(x)| \leq 2^{-n}$. Thus, $f_n^{(m)} \nearrow u_m$.

Proof (c): Note that $u = \sum_{m=0}^{\infty} u_m$, thus by Corollary 9.9, we have

$$\int u\, d\lambda = \sum_{m=0}^{\infty} \int u_m\, d\lambda.$$

We calculate first $\int u_m \, d\lambda$ using Corollary 9.7

$$\begin{aligned}
\int u_m \, d\lambda &= \lim_{n\to\infty} \int f_n^{(m)} \, d\lambda \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{1}{2^m}\Big(1 + \frac{k}{2^n}\Big) \cdot \lambda\Big(\Big[m + \frac{k}{2^n}, m + \frac{k+1}{2^n}\Big)\Big) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{1}{2^m}\Big(m + \frac{k}{2^n}\Big) \cdot \frac{1}{2^n} \\
&= \lim_{n\to\infty} \frac{1}{2^m}\Big(m + \frac{1}{4^n} \cdot \sum_{k=0}^{2^n-1} k\Big) \\
&= \lim_{n\to\infty} \frac{1}{2^m}\Big(m + \frac{1}{4^n} \cdot \frac{2^n(2^n-1)}{2}\Big) \\
&= \frac{1}{2^m}\Big(m + \frac{1}{2}\Big) \\
&= \frac{2m+1}{2^{m+1}}.
\end{aligned}$$

Using this we have,

$$\begin{aligned}
\int u \, d\lambda &= \sum_{m=0}^{\infty} \int u_m \, d\lambda \\
&= \sum_{m=0}^{\infty} \frac{2m+1}{2^{m+1}} \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{2^{m-1}} + \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{2^m} \\
&= 3.
\end{aligned}$$