Universiteit Utrecht — Boedapestlaan 6
Mathematisch Instituut — 3584 CD Utrecht

Measure and Integration: Mid-Term, 2019-20

(1) Let X be a set. We call collection $\mathcal{F}$ of subsets of X an algebra if the following conditions hold:

(i) $\emptyset \in \mathcal{F}$, (ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and (iii) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

(a) Let $\mathcal{F}_1 \subset \mathcal{F}_1 \subset \cdots$ be a strictly increasing sequence of algebras on X. Show that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is an algebra on X. (1 pt)

(b) Let μ be a **pre**-measure on $\bigcup_{n=1}^{\infty} \mathcal{F}_n$. Find a measure ν on $\sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{F}_n\Big)$ extending μ, i.e. $\mu(A) = \nu(A)$ for all $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. (1 pts)

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider the collection

$$\mathcal{C} = \Big\{ C \subseteq X : \forall \epsilon > 0, \exists A_\epsilon, B_\epsilon \in \mathcal{A} \text{ such that } A_\epsilon \subseteq C \subseteq B_\epsilon \text{ and } \mu(A_\epsilon \Delta B_\epsilon) < \epsilon \Big\}.$$

(a) Prove that $\mathcal{C}$ is a σ-algebra on X containing $\mathcal{A}$. (2 pts)

(b) Define ν on $\mathcal{C}$ by $\nu(C) = \inf \Big\{ \mu(B) : C \subseteq B \text{ and } B \in \mathcal{A} \Big\}$. Prove that ν is an **outer** measure on $\mathcal{C}$ extending μ on $\mathcal{A}$, i.e. $\nu(\emptyset) = 0$, ν is monotone and σ-subadditive, and $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$. (2 pts)

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ is Lebesgue measure. Define a function $u : \mathbb{R} \to \mathbb{R}$ by $u(x) = \sum_{n=0}^{\infty} \frac{x}{2^n} \cdot \mathbb{I}_{[n,n+1)}(x)$.

(a) Prove that u is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\overline{\mathbb{R}})$-measurable. (0.5 pts)

(b) For $m \geq 0$, let $u_m(x) = \frac{x}{2^m} \cdot \mathbb{I}_{[m,m+1)}(x)$. Give an explicit sequence $(f_n^{(m)})_{n \in \mathbb{N}}$ of $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable simple functions with $f_n^{(m)} \nearrow u_m$, i.e. $(f_n^{(m)})$ is an increasing sequence and $u_m(x) = \sup_{n \geq 1} f_n^{(m)}(x) = \lim_{n \to \infty} f_n^{(m)}(x)$ for all $x \in \mathbb{R}$. (1.5 pts)

(c) Use part (b) to prove that

$$\int u \, d\lambda = \sum_{n=0}^{\infty} \frac{2n+1}{2^{n+1}} = 3.$$

(Recall that $\sum_{n=0}^{m} k = \frac{m(m+1)}{2}$, $\sum_{n=0}^{\infty} r^n = \frac{1}{r-1}$ and $\sum_{n=1}^{\infty} n r^{n-1} = \frac{1}{(r-1)^2}$, for $-1 < r < 1$) (2 pts)