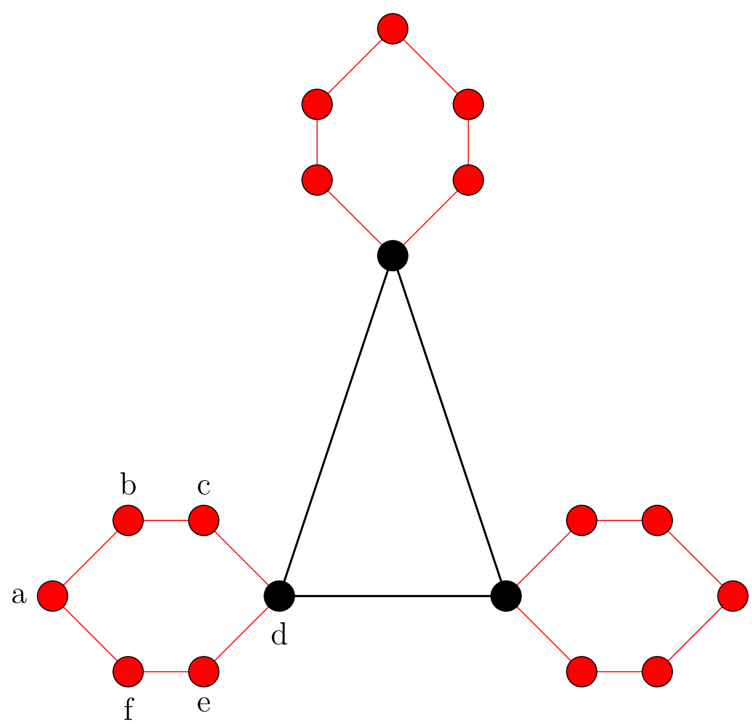
Methods and Models in Complex Systems BETA-B2-CS – Exam solutions

November 6, 2019

1. (a) The network has been built from rings because if one link breaks (e.g. due to digging) the nodes are still connected. This provides robustness at a reasonable cost.

   (b) A drawing of the graph for $n_f = 3$ and $n_c = 5$, where a black circle is a distribution point, and a red circle is a home:



(c) The number of edges is n_f for the fiber optic links and $n_f(n_c+1)$ for the copper links, giving a total of $n_f(n_c+2)$.

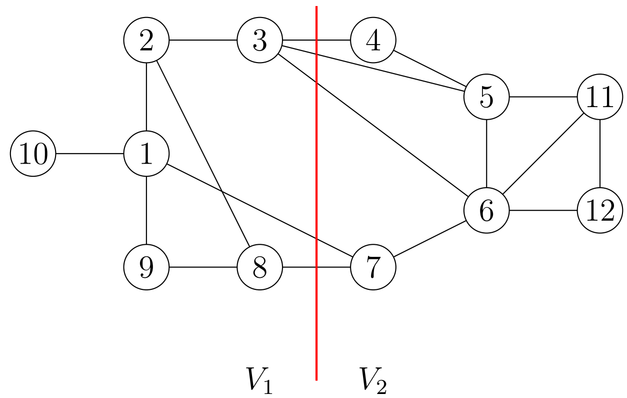
(d) The diameter of a ring with n_r nodes is $n_r/2$ if n_r is even, and $(n_r-1)/2$ if n_r is odd. To get from the farthest home in a copper ring to its distribution point we have to traverse $(n_c+1)/2$ links (because n_c is odd and hence n_c+1 is even). To get from this distribution point to the distribution point farthest away, we traverse $(n_f-1)/2$ links. And then to reach the farthest home from that distribution point we need to traverse another $(n_c+1)/2$ links. Therefore, the diameter of the whole network is $n_c+1+(n_f-1)/2 = n_c+(n_f+1)/2$.

(e) If we cut one of the fiber optic links the diameter of the main ring increases from $(n_f-1)/2$ to n_f-1, i.e. an extra $(n_f-1)/2$ links have to be traversed. The maximum distance from a home to its distribution point is not affected. The diameter of the whole network hence increases to n_c+n_f.

(f) A vertex d can lie on a path from 5 homes of its copper ring to the 12 nodes of the other rings, giving 60 paths. It can also lie on 1 out of 2 shortest paths (b,e), the unique shortest path (c,e), and 1 out of 2 shortest paths (c,f), contributing together 2 to the betweenness centrality. Altogether this gives 62. We multiply this by a factor of 2 if we count the reverse directions as well. Thus $C_B(d) = 124$.

(g) The home farthest away from d is a, and it has a centrality 2, only from paths within the copper ring. Counting reverse directions this gives $C_B(a) = 4$. Note that we could also ignore the reverse directions, which would normalize C_B differently but would not change the relative importance of the vertices.

2. The graph can be drawn as follows, with a red line showing the partitioning:



(a)

(b) The edge cut is the number of edges with one end vertex in V_1 and another in V_2, i.e. the edges: (3,4), (3,5), (3,6), (1,7), (7,8). So the edge cut is 5.

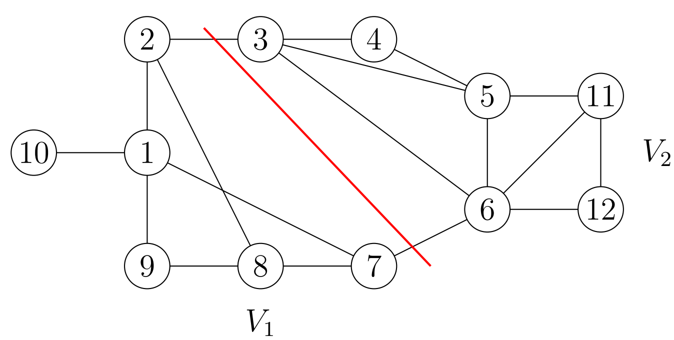
(c) [2 pt] The modularity M of a clustered network can be computed by

$$M = \sum_{c=1}^{n_c} \left[\frac{L_c}{L} - \left(\frac{k_c}{2L} \right)^2 \right] = \frac{1}{L} \sum_{c=1}^{n_c} \left[L_c - \frac{k_c^2}{4L} \right].$$

Here, $n_c = 2$, $L = 18$, $L_1 = 6$, $L_2 = 7$, $k_1 = 17$, $k_2 = 19$. Therefore,

$$M = \frac{1}{18} \left[6 - \frac{17^2}{72} + 7 - \frac{19^2}{72} \right] = \frac{1}{18} \left[13 - \frac{650}{72} \right] \approx \frac{4}{18} \approx 0.22$$

(d) Swapping vertices 3 and 7 reduces the edge cut from 5 to 2. This an be shown in the same picture by a different split:



(e) For the new partitioning/clustering, $n_c = 2$, $L = 18$, $L_1 = 7$, $L_2 = 9$, $k_1 = 16$, $k_2 = 20$.

Therefore,

$$M = \frac{1}{18}\left[7 - \frac{16^2}{72} + 9 - \frac{20^2}{72}\right] = \frac{1}{18}\left[16 - \frac{656}{72}\right] \approx \frac{7}{18} \approx 0.38$$

3. (a)

$$\begin{cases} \frac{d}{dt}s(t) = \delta x(t) - \beta s(t)x(t) \\ \frac{d}{dt}x(t) = -\delta x(t) + \beta s(t)x(t) \\ x(t)+s(t) = 1, \end{cases}$$

subjected to initial conditions $s(0) = s_0,\ x(0) = 1 - s_0$.

(b) First, let us use the equality and write: $s(t) = 1 - x(t)$. The system simplifies to a single differential equation:

$$\frac{d}{dt}x(t) = -\delta x(t) + \beta(1 - x(t))x(t)$$

(c) having two fixed points:

$$-\delta x^* + \beta(1 - x^*)x^* = 0 \implies x^* = 0 \text{ or } x^* = 1 - \delta/\beta$$

Therefore we have two sets of fixed points:
1. FP1: $s^* = 1,\ x^* = 0$.
2. FP2: $s^* = \delta/\beta,\ x^* = 1 - \delta/\beta$.
By differentiating the right hand side of the ODE we obtain,

$$\frac{\partial}{\partial x}(-\delta x + \beta(1 - x)x) = -\delta + \beta - 2\beta x.$$

Therefore:
1. FP1: $-\delta + \beta - 2\beta x\Big|_{x=0} = \beta - \delta$. When $\beta > \delta$ FP1 is unstable, when $\beta < \delta$, FP1 is stable – everyone will recover eventually.
2. FP2: $-\delta + \beta - 2\beta x\Big|_{x=1-\delta/\beta} = \delta - \beta$. When $\beta > \delta$, FP2 is stable – there will always be a positive fraction of infected individuals , when $\beta < \delta$ FP2 is unstable.

(d) By elaborating a little more on the answer to (b) we have

$$\frac{d}{dt}x(t) = -\delta x(t) + \beta(1 - x(t))x(t)\Big|_{\delta=0} = \beta(1 - x(t))x(t)$$

We obtained the logistic equation. Logistic equation is typically used to study growth of a population with limited resources. In the current case the recovery rate is zero, so the infection will spread until it takes over all individuals. That is to say, individuals play role of the limited resource.

4. Permanent immunity means that individuals are recovered and cannot be infected again, therefore, we need to introduce a new species R.

$$\begin{cases} \frac{d}{dt}s(t) = -\beta s(t)x(t) \\ \frac{d}{dt}x(t) = -\mu x(t) + \beta s(t)x(t) \\ \frac{d}{dt}r(t) = \mu x(t) \\ x(t)+s(t) + r(r) = 1, \end{cases} \tag{1}$$

subjected to initial conditions $s(0) = s_0,\ x(0) = 1 - s_0$.

5. (a) we order the cities as follows (Paris, Milan, Amsterdam), then the right-stochastic transition matrix has the following form:

$$\mathbf{M} = \begin{bmatrix} 0 & 2/3 & 1/2 \\ 2/3 & 0 & 1/2 \\ 1/3 & 1/3 & 0 \end{bmatrix}$$

(b) The state vector for X being in Paris with probability one is given by: $\mathbf{s_0} = (1, 0, 0)^\top$, then
$\mathbf{s_1} = \mathbf{M}s_0 = (0, 2/3, 1/3)^\top$;
In two days, $\mathbf{s_2} = \mathbf{M}s_1 = (11/18, 1/6, 2/9)^\top$ – probability that X is in Paris is 11/18.

(c) We are interested in a fixed point:

$$\mathbf{M}x = x,$$

where $\mathbf{x} = (a, b, c)^\top$. We may rewrite this equation as

$$\begin{bmatrix} -1 & 2/3 & 1/2 \\ 2/3 & -1 & 1/2 \\ 1/3 & 1/3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Notice the symmetry: If you swap Paris and Milan the transition matrix does not change. Therefore $a = b$. On another hand $a + b + c = 1$ so that $c = 1 - 2a$. This tells us that our solution should be of the form: $\mathbf{x} = (a, a, 1 - 2a)$. By multiplying with, for instance, the first raw of $\mathbf{M}$ we obtain:

$$(-1, \frac{2}{3}, \frac{1}{2})(a, a, 1 - 2a)^\top = 0 \implies -a + \frac{2}{3}a + \frac{1}{2}(1 - 2a) = 0$$

and therefore $a = \frac{3}{8}$ and $\mathbf{x} = (\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$.