Methods and Models in Complex Systems
BETA-B2-CS – Retake Exam

January 9, 2020

(Time: 3 hours. Please motivate your answers and simplify your results where possible.)

1. Consider a social network that consists of a set of r communities. Each community is a clique consisting of k persons. One person in each clique is considered an *influencer*. All influencers are connected in a ring. Assume for convenience that r and k are odd.

   (a) [2 pt] Draw the corresponding graph for $r = 3$ and $k = 5$.

   (b) [1 pt] How many edges does the graph possess for general r and k?

   (c) [2 pt] What is the diameter of the graph for general r and k?

   (d) [4 pt] Determine the betweenness centrality of each person in the social network for $r = 3$ and $k = 5$.

   (e) [1 pt] What is the special role of the influencers when information flows through the network?

2. We build a *donut* (or torus) network as follows. Given is a rectangular grid of size $k_x \times k_y$ consisting of vertices (i, j) with $1 \leq i \leq k_x$ and $1 \leq j \leq k_y$. Each vertex (i, j) is connected to its 4 neighbours $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, and $(i, j-1)$. At the boundaries of the grid, the connecctions wrap around, so that (i, k_y) is connected to $(i, 1)$ and (k_x, j) is connected to $(1, j)$. Assume for convenience that k_x and k_y are even and that $k_x, k_y \geq 2$.

   (a) [2 pt] Draw the graph for $k_x = 4, k_y = 8$ with k_x points in the horizontal direction.

   (b) [2 pt] Draw two different splittings, horizontal and vertical, each into two clusters of equal size. What is the edge cut of the horizontal splitting? And of the vertical splitting?

   (c) [3 pt] The modularity M of a clustered network can be computed by

   $$M = \sum_{c=1}^{n_c} \left[\frac{L_c}{L} - \left(\frac{k_c}{2L} \right)^2 \right].$$

   Here, n_c is the number of clusters (communities), L is the total number of links in the network, L_c is the number of links within community c, and k_c is the sum of the degrees of the nodes in community c.

   What is the modularity of the clustering into two clusters based on the given horizontal split of the 4×8 donut network? Give the answer as a fraction, simplified as much as possible. What is the modularity for the vertical split?

   (d) [3 pt] What is the best modularity we can achieve for a fixed total number n of grid points, where we are free to choose the aspect ratio k_x/k_y?

3. Consider a modification of the SIR model called SIRS. The model is defined by the following rules:

$$S + I \xrightarrow{\alpha} 2I$$

$$I \xrightarrow{\beta} R$$

$$R \xrightarrow{\delta} S$$

where $\alpha, \beta, \delta > 0,\ \alpha \neq \beta$ are the rates. Let $s(t)$, $x(t)$, and $r(t)$ denote concentration of correspondingly S, I and R species, with $s(t) + x(t) + r(t) = 1$

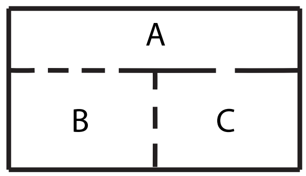
(a) [2 pt] Formulate the system of ordinary differential equations for $s(t)$, $x(t)$ and $r(t)$.

(b) [2 pt] Show that this system can be well-represented by two differential equation for $s(t), x(t)$, write down these equations.

(c) [2 pt] Write down Jacobian matrix for this system of ODEs.

(d) [4 pt] Find all fixed points of the form (s^*, x^*), classify their stability depending on the parameters.

4. A trained mouse lives in the house shown. A bell rings at regular intervals, and the mouse is trained to change rooms (A,B, or C) each time it rings. When it changes rooms, it is equally likely to pass through any of the doors in the room it is in.



(a) [3 pt] Write down the right-stochastic transition probability matrix.

(b) [3 pt] Suppose the mouse is in room C, what is the probability it will again be in room C after the bell rings once, two times?

(c) [4 pt] Given this process goes on infinitely long, what fraction of its life will the mouse spend in each room?