Exam Stochastic processes - WISB362 - Spring 2018

Utrecht University - Monday 25 June 2018, 17:00h - 20:00h

Total number of points: 100.
***NO** books or handwritten material are allowed.*

***Exercise* 1**

Suppose that people arrive at a bus stop in accordance with a Poisson process with rate λ (per hour). Let T_n denote the nth interarrival time and S_n the time of the nth arrival. Let $N(t)$ denote the number of arrivals by time t.

(a) **[6 points]** Suppose exactly one passenger has arrived by time t. Prove that the distribution of the time at which this event occurred is Uniform($[0,t]$). In other words, show that, for $s \leq t$,

$$\mathbb{P}\{T_1 < s \,|\, N(t) = 1\} = \frac{s}{t}.$$

The bus departs at time t. Let X denote the total amount of waiting time of all those who get on the bus at time t.

(b) **[5 points]** Compute $\mathbb{E}[X|N(t)]$.

(c) **[6 points]** Compute $\mathbb{P}\{N(t) = 10, N(t/2) = 5 \,|\, N(t/4) = 3\}$.

(d) **[6 points]** Suppose that for a bus stop in a little village the Poisson rate equals $\lambda = 6$. Show that the probability that there are 8 or more arrivals in an hour, given there were 2 arrivals in the first 10 minutes, equals

$$1 - \sum_{y=0}^{5} e^{-5} \frac{5^y}{y!}.$$

(e) **[6 points]** Assume females and males arrive at times of independent Poisson processes with rates 2 and 4 passengers per hour, respectively. What is the probability the first two arrivals are female passengers?

***Exercise* 2**

Let $\{X(t), t \geq 0\}$ be a standard Brownian motion process.

(a) **[6 points]** Determine the distribution of $X(s) + X(t)$, $s \leq t$.

(b) **[6 points]** Determine $\mathbb{P}\{X(5) < X(1)\}$.

***Exercise* 3**

The general (finite) random walk is defined as follows. The states are numbered $s_0, s_1, \ldots, s_n$. If the process is in s_i, then it moves to s_{i-1} with probability q_i, it stays in s_i with probability r_i, and moves to s_{i+1} with probability p_i. (Where $p_i + q_i + r_i = 1$, $q_0 = 0$, $p_n = 0$.)

(a) [**4 points**] Let $P_{(n)}$ be the transition probability matrix corresponding to the general random walk with $n+1$ states. Write down the transition probability matrix $P_{(2)}$.

(b) [**7 points**] Let $\pi = [\pi_0, \ldots, \pi_n]$ be the vector containing the long-run proportions of time that the random walk is in a specific state. Show, by mathematical induction, that $\pi_{i+1} q_{i+1} = \pi_i p_i$.

(c) [**8 points**] Suppose $n = 2$ and $r_i = 0$ for $i = 0, 1, 2$. Let f_0 be the probability that, starting in state s_0, the process will return to this state at some step. Show that $f_0 = 1$.

(d) [**7 points**] Consider the above described random walk with states $s_0, \ldots, s_n$ for which $r_i = 0$ for all $i \in \{0, \ldots, n\}$ and $p_i = p$ for all $i \in \{1, \ldots, n-1\}$. Show that this is a time reversible process for $p \in (0, 1)$.

***Exercise* 4**

A service center consists of two servers, each working at an exponential rate of two services per hour. Customers arrive at a Poisson rate of three per hour. Assume that the system capacity is at most three customers.

(a) [**4 points**] With the number of customers in the system as the state, find the birth and death rates of the process.

(b) [**6 points**] Show that $P_0 = \frac{32}{143}$, where P_j should be interpreted as the long-run proportion of time that the process is in state j.

(c) [**5 points**] What fraction of potential customers enter the system?

(d) [**6 points**] What would the values of part (b) and (c) be if there was only a single server, and his rate was twice as fast (that is, $\mu = 4$)?

***Exercise* 5**

A set of n fair dice is thrown. All those that land on six are put aside, and the others are again thrown. This is repeated until all the dice have landed on six. Let N denote the number of throws needed. (For instance, suppose that $n = 3$ and that on the initial throw exactly two of the dice land on six. Then the other die will be thrown, and if it lands on six, then $N = 2$.) Let $m_n = \mathbb{E}[N]$.

(a) [**6 points**] Show that the recursive formula for m_n equals

$$m_n = \frac{1 + \sum_{i=1}^{n-1} m_{n-i} \binom{n}{i} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{n-i}}{1 - \left(\frac{5}{6}\right)^n}.$$

Hint: Use conditioning on Y: the number of dice that land on six in the first roll.

(b) [**6 points**] Let X_i denote the number of dice rolled on the ith throw. Find

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i\right].$$

End of exam

Main probabilistic laws

Discrete laws:

Name	Ω	**Probability mass**	**Mean**	**Variance**
Bernoulli(p)	$\{a,b\}$	$p(b) = p = 1-p(a)$	$a+p(b-a)$	$p(1-p)(a-b)^2$
Binomial(n,p)	$\{0,1,\dots,n\}$	$p(k) = \binom{n}{k} p^k(1-p)^{n-k}$	np	$np(1-p)$
Geometric(p)	$\mathbb{N}_{\geq 1}$	$p(k) = (1-p)^{k-1}p$	$1/p$	$(1-p)/p^2$
Poisson(λ)	$\mathbb{N}$	$p(k) = \mathrm{e}^{-\lambda}\lambda^k/k!$	λ	λ

Continuous laws:

Name	**Probability density**	**Mean**	**Variance**
Unifom(a,b)	$f(x) = \begin{cases} 1/(b-a) & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$(a+b)/2$	$(b-a)^2/12$
Exponential(λ)	$f(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$1/\lambda$	$1/\lambda^2$
Normal(μ,σ^2)	$f(x) = (1/(\sqrt{2\pi}\,\sigma)\exp\big\{-(x-\mu)^2/(2\sigma^2)\big\}$	μ	σ^2